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Periodic Wavelets in Walsh Analysis

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Abstract. The main aim of this paper is to present a review of periodic wavelets related to the generalized Walsh functions on the p-adic Vilenkin group G_p. In addition, we consider several examples of wavelets in the spaces of periodic complex sequences. The case $p = 2$ corresponds to periodic wavelets associated with the classical Walsh functions.

1. Introduction

Let $\mathbb{Z}_p$ be the discrete cyclic group of order p, i.e., the set $\{0,1,\dots,p\}$ with the discrete topology and modulo p addition. The *p-adic Vilenkin group G* is defined to be the subgroup of $\prod_{j\in\mathbb{Z}}\mathbb{Z}_p$ consisting of sequences

$$x=(x_j)=(\dots,x_{-2},x_{-1},x_0,x_1,x_2,\dots),$$

for which there exists $k=k(x)\in\mathbb{Z}$ such that $x_j=0$ for all $j<k$. The group operation on G is denoted by $\oplus$ and defined as the coordinate-wise addition modulo p:

$$(z_j)=(x_j)\oplus(y_j) \iff z_j=x_j+y_j(\operatorname{mod} p) \quad \text{for all } j\in\mathbb{Z}.$$

Let us denote the inverse operation of $\oplus$ by $\ominus$ (so that $x\ominus x=\theta$, where θ is the zero sequence). One can put a topology on G as the product topology inherits from $\prod_{j\in\mathbb{Z}}\mathbb{Z}_p$. The group G is a locally compact abelian group and the sets

$$U_l:=\{(x_j)\in G|\ x_j=0 \ \text{for } j\le l\},\quad l\in\mathbb{Z},$$

form a complete system neighbourhoods of the zero sequence. Notice also that

$$U_{l+1}\subset U_l \ \text{ for } l\in\mathbb{Z},\quad \bigcap U_l=\{\theta\},\quad \bigcup U_l=G.$$

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One can show that G is self-dual. The duality pairing on G takes $x=(x_j)$ and $\omega=(\omega_j)$ to

$$\chi(x,\omega)=\exp\left(\frac{2\pi i}{p}\sum_{j\in\mathbb{Z}}x_j\omega_{1-j}\right).$$

Consider $U=U_0$ as a subgroup of G. This subgroup, when $p=2$, is isomorphic to the *Cantor group*, which is the topological Cartesian product of countably many cyclic groups of order 2 with discrete topology. It is well-known that U is a perfect nowhere-dense totally disconnected metrizable space and, therefore, U is homeomorphic to the Cantor ternary set (e.g., [6, Chapter 14]). There exists a Haar measure on G normalized so that the measure of U is 1. For simplicity, we shall denote this measure by dx.

As usual, the Lebesgue space $L^2(G)$ consists of all square integrable functions on G. For each function $f\in L^1(G)\cap L^2(G)$, its Fourier transform $\widehat{f}$,

$$\widehat{f}(\omega)=\int_G f(x)\overline{\chi(x,\omega)}dx,\quad \omega\in G,$$

belongs to $L^2(G)$. The Fourier operator

$$\mathscr{F}:L^1(G)\cap L^2(G)\to L^2(G),\quad \mathscr{F}f=\widehat{f},$$

extends uniquely to the whole space $L^2(G)$. See [22] and [33] for further details about harmonic analysis on the group G.

Consider the mapping $\lambda:G\to\mathbb{R}_+$ defined by

$$\lambda(x)=\sum_{j\in\mathbb{Z}}x_jp^{-j},\quad x=(x_j)\in G.$$

Take in G a discrete subgroup $H=\{(x_j)\in G|x_j=0 \text{ for } j>0\}$. The image of the subgroup H under λ is the set of non-negative integers: $\lambda(H)=\mathbb{Z}_+$. For each $k\in\mathbb{Z}_+$, let $h_{[k]}$ denote the element of H such that $\lambda(h_{[k]})=k$ (clearly, $h_{[0]}=\theta$). The *generalized Walsh functions* on G can be defined by

$$w_k(x)=\chi(x,h_{[k]}),\quad x\in G,\ k\in\mathbb{Z}_+.$$

So, these functions are characters for G. Also, it is well-known that $\{w_k|\ k\in\mathbb{Z}_+\}$ is an orthonormal basis for $L^2(U)$ (when $p=2$, we have the classical Walsh system).

Using the elements of H as translations, one can study wavelets in $L^2(G)$. Orthogonal wavelets and refinable functions representable as lacunary Walsh series were introduced for the first time by Lang [24] in the context of the Cantor dyadic group and, subsequently, they have been extended and studied by several authors (see, e.g., [7]-[19], [31], [32], [37], [38]). Multiresolution analysis of functions defined on the Cantor dyadic group was studied independently by Bl. Sendov ([34]-[36]). Wavelets on the p-adic Vilenkin group G by means of an iterative method giving rise to so-called wavelet sets were derived by J.J. Benedetto and R.L. Benedetto [2]. At the same time, an approach developed in [2] can be applied to wavelets on the additive group of p-adic numbers (cf. [1], [23], [25], [39]).

This paper is a continuation of our review [18], where among the main subjects are the following:

- algorithms to construct orthogonal and biorthogonal wavelets associated with the Walsh polynomials;
- estimates of the smoothness of dyadic orthogonal wavelets of Daubechies type;
- an algorithm for constructing Parseval dyadic frames.

The aim of this paper is to present a review of periodic wavelets related to the generalized Walsh functions. In Section 2, by analogy with the periodic wavelets on the line $\mathbb{R}$ (see, e.g., [4], [5], [20], [27]-[30], [40], [41]), we define periodic wavelets on G and consider the corresponding algorithms for decomposition and reconstruction. Similar results for the case $p = 2$ are given in the recent papers [11] and [19]. Then, in Section 3, we use the generalized Walsh functions to define wavelets in the space $\mathbb{C}_N$ consisting of all sequences $x = (\dots, x(-1), x(0), x(1), x(2), \dots)$, such that $x(j+N) = x(j)$ for all $j \in \mathbb{Z}$ (cf. [3], [13], [21], [29]).

2. Periodic wavelets on the p-adic Vilenkin group

To keep our notation simple, we write $N := p^n$ and $\varepsilon_p := \exp(2\pi i/p)$. Define an automorphism $A \in \operatorname{Aut} G$ by the formula $(Ax)_j = x_{j+1}$ for all $x = (x_j) \in G$. Then, for $0 \le k \le N-1$, we let $x_{n,k} := A^{-n} h_{[k]}$ and $U_k^{(n)} := x_{n,k} + A^{-n}(U)$. It is easily seen that the sets $U_k^{(n)}$ are cosets of the subgroup $A^{-n}(U)$ in the group U, and that

$$U_k^{(n)} \cap U_l^{(n)} = \emptyset \quad \text{for } k \neq l, \qquad \bigcup_{k=0}^{N-1} U_k^{(n)} = U.$$

Moreover, it is clear that $w_l(x)$ with $0 \le l \le N-1$ is constant on $U_k^{(n)}$ for each $0 \le k \le N-1$. We shall use the notation

$$w_{l,k}^{(n)} := w_l(x_{n,k}) \quad \text{for } \ 0 \le l, k \le N-1.$$

Notice that

$$w_{l,k}^{(n)} = w_{k,l}^{(n)} = \varepsilon_p^{-sq} w_{pk+s,Nq+l}^{(n+1)}, \quad 0 \le s, q \le p-1, \tag{2.1}$$

$$\sum_{i=0}^{N-1} w_{i,l}^{(n)} \overline{w_{i,k}^{(n)}} = \sum_{j=0}^{N-1} w_{l,j}^{(n)} \overline{w_{k,j}^{(n)}} = N\delta_{l,k}, \quad 0 \le l, k \le N-1. \tag{2.2}$$

A finite sum

$$D_N(x) := \sum_{j=0}^{N-1} w_j(x), \quad x \in G,$$

is called the *Walsh-Dirichlet kernel* of order N. It is well-known that

$$D_N(x) = \begin{cases} N, & x \in U_0^{(n)}, \\ 0, & x \in U \setminus U_0^{(n)}. \end{cases}$$

Let us introduce the following spaces

$$V_n := \mathrm{span}\{1, w_1(x), \dots, w_{N-1}(x)\},$$

$$W_n^{(j)} := \mathrm{span}\{w_{jN}(x), w_{jN+1}(x), \dots, w_{(j+1)N-1}(x)\},$$

where $j = 1, \dots, p-1$. Note that the orthogonal direct sum of V_n, $W_n^{(1)}, \dots, W_n^{(p-1)}$ coincides with V_{n+1}, that is, for $W_n := W_n^{(1)} \bigoplus \cdots \bigoplus W_n^{(p-1)}$, we have $V_n \bigoplus W_n = V_{n+1}$. The spaces V_n and $W_n^{(j)}$ will be called the *approximation spaces* and *wavelet spaces*, respectively.

We can use the discrete Vilenkin-Chrestenson transform to recover $v \in V_n$ from the values $v(x_{n,l})$, $0 \le l \le N-1$. Indeed, if

$$v(x) = \sum_{k=0}^{N-1} c_k w_k(x), \quad x \in U, \tag{2.3}$$

then

$$c_k = \frac{1}{N} \sum_{l=0}^{N-1} v(x_{n,l}) \overline{w_{l,k}^{(n)}}, \quad 0 \le k \le N-1; \tag{2.4}$$

see, e.g., [22, Section 11.2], where the corresponding fast algorithm is given.

Suppose that $a = (a_0, a_1, \dots, a_{N-1})$, where $a_k \ne 0$, $0 \le k \le N-1$. Then we set

$$\Phi_N^a(x) := \frac{1}{N} \sum_{k=0}^{N-1} a_k w_k(x), \;\; \varphi_{n,k}(x) := \Phi_N^a(x \ominus x_{n,k}), \quad 0 \le k \le N-1, \; x \in G.$$

Proposition 2.1. *Let $v \in V_n$. Assume that*

$$\alpha_{n,k} = \alpha_{n,k}(v) := \sum_{l=0}^{N-1} a_l^{-1} c_l w_{l,k}^{(n)}, \quad 0 \le k \le N-1, \tag{2.5}$$

where c_l are defined as in (2.4). *Then*

$$v(x) = \sum_{k=0}^{N-1} \alpha_{n,k} \, \varphi_{n,k}(x). \tag{2.6}$$

Proof. According to (2.2), for any $v \in V_n$ we get

$$\sum_{k=0}^{N-1} w_{l,k}^{(n)} \varphi_{n,k}(x) = a_l w_l(x), \quad 0 \le l \le N-1,$$

and, in view of (2.3), (2.4) and (2.5),

$$v(x) = \sum_{l=0}^{N-1}\sum_{j=0}^{N-1} a_l^{-1} c_l w_{l,j}^{(n)} \varphi_{n,j}(x) = \sum_{k=0}^{N-1} \alpha_{n,k}\, \varphi_{n,k}(x).$$

Therefore, the expansion in (2.6) is valid for all $v \in V_n$. □

Remark 2.1 (cf. [40, Proposition 9]). Suppose that $\widetilde{\varphi}_{n,k}$ are defined by

$$\widetilde{\varphi}_{n,0}(x) = \sum_{j=0}^{N-1} \overline{a}_j^{-1} w_j(x),\ \widetilde{\varphi}_{n,k}(x) = \widetilde{\varphi}_{n,0}(x \ominus x_{n,k}), \quad k = 1, \dots, N-1.$$

Then $\{\widetilde{\varphi}_{n,k}\}_{k=0}^{N-1}$ is a dual shift basis for $\{\varphi_{n,k}\}_{k=0}^{N-1}$. Indeed, using (2.3) and (2.5), for any $v \in V_n$ we have

$$\begin{aligned}
(v, \widetilde{\varphi}_{n,k}) &:= \int_U v(x), \overline{\widetilde{\varphi}_{n,k}(x)} dx \\
&= \int_U \Big(\sum_l c_l w_l(x)\Big) \overline{\widetilde{\varphi}_{n,0}(x \ominus x_{n,k})} dx \\
&= \int_U \Big(\sum_l c_l w_l(x)\Big) \Big(\overline{\sum_l \overline{a_l^{-1} w_{l,k}^{(n)}} w_l(x)}\Big) dx \\
&= \alpha_{n,k}(v),
\end{aligned}$$

where the last equality follows from the orthogonality of the system $\{w_k \mid k \in \mathbb{Z}_+\}$.

Let $b = (b_0, b_1, \dots, b_{pN-1})$, where $b_k \neq 0$ for all $0 \le k \le pN-1$. In particular, we can choose

$$b_k = \begin{cases} a_{k/p} & \text{if } k \text{ is divisible by } p, \\ 1 & \text{if } k \text{ is not divisible by } p \end{cases} \qquad \text{or} \qquad b_k = \begin{cases} a_k & \text{if } k \le N-1, \\ 1 & \text{if } 0 \le k \le pN-1. \end{cases}$$

We set

$$\varphi_{n+1,k}(x) := \Phi_{pN}^b(x \ominus x_{n+1,k}), \quad 0 \le k \le pN-1,$$

where

$$\Phi_{pN}^b(x) := \frac{1}{pN} \sum_{k=0}^{pN-1} b_k w_k(x), \quad x \in G.$$

Then we define

$$\psi_{n,k}^{(j)}(x) := \sum_{s=0}^{p-1} \varepsilon_p^{js}\, \varphi_{n+1,pk+s}(x), \quad 0 \le k \le N-1,\ 1 \le j \le p-1.$$

Let us show that, for each j, the system $\{\psi_{n,k}^{(j)}\}_{k=0}^{N-1}$ is a bases for the corresponding wavelet space $W_n^{(j)}$.

Proposition 2.2. *Suppose that $w \in W_n^{(j)}$ for some $j \in \{1, \dots, p-1\}$. Then*

$$w(x) = \sum_{k=0}^{N-1} \beta_{n,k}^{(j)} \psi_{n,k}^{(j)}(x), \tag{2.7}$$

where, with the notation as in (2.4)*,*

$$\beta_{n,k}^{(j)} = \beta_{n,k}^{(j)}(w) = \sum_{l=0}^{N-1} b_{jN+l}^{-1} c_{jN+l} w_{jN+l,pk}^{(n+1)}, \quad 0 \le k \le N-1. \tag{2.8}$$

Proof. Let $w \in W_n^{(j)}$ where $j \in \{1, \dots, p-1\}$. Then, since $W_n^{(j)} \subset V_{n+1}$, as in Proposition 2.1 we have

$$\begin{aligned} w(x) &= \sum_{l=jN}^{(j+1)N-1} c_l w_l(x) \\ &= \sum_{k=0}^{pN-1} \alpha_{n+1,k}(w)\, \varphi_{n+1,k}(x) \\ &= \sum_{s=0}^{p-1} \sum_{k=0}^{N-1} \alpha_{n+1,pk+s}(w)\, \varphi_{n+1,pk+s}(x), \end{aligned} \tag{2.9}$$

where

$$\alpha_{n+1,pk+s}(w) = \sum_{l=0}^{N-1} b_{jN+l}^{-1} c_{jN+l} w_{jN+l,pk+s}^{(n+1)},$$

$$c_{jN+l} = \frac{1}{pN} \sum_{l=0}^{pN-1} w(x_{n+1,l}) \overline{w_{l,jN+l}^{(n+1)}}.$$

Here, in view of (2.1), $w_{jN+l,pk+s}^{(n+1)} = \varepsilon_p^{js} w_{jN+l,pk}^{(n+1)}$, and hence

$$\alpha_{n+1,pk+s}(w) = \varepsilon_p^{js} \alpha_{n+1,pk}(w), \quad 0 \le k \le N-1,\ 0 \le s \le p-1,$$

which by (2.8) and (2.9) yields (2.7). □

Let $\alpha \neq 0$. Propositions 2.1 and 2.2 for the case where

$$a_k = \begin{cases} \alpha & \text{if } k = 0 \text{ or } k = N-1, \\ 1 & \text{otherwise} \end{cases} \tag{2.10}$$

can be found in [15]. In this case, we set

$$b_k = \begin{cases} \alpha & \text{if } k = 0 \text{ or } k = pN-1, \\ 1 & \text{otherwise} \end{cases}$$

Note that the value $\alpha = 1$ corresponds to the Haar wavelets (so, we use $\alpha \neq 1$ in the sequel).

For each $l \in \{0, 1, \dots, N-1\}$ with p-ary expansion

$$l = \sum_{j=0}^{n-1} \nu_j p^j, \quad \nu_j \in \{0, 1, \dots, p-1\},$$

we let $\gamma(l) := \sum_{j=0}^{n-1} \nu_j$. According to [15], in the case (2.10) we have the following equalities

$$\varphi_{n,k}(x) = \sum_{s=0}^{p-1} \varphi_{n+1,pk+s}(x) - \frac{(1-\alpha)}{N} \varepsilon_p^{-\gamma(k)} w_{N-1}(x), \tag{2.11}$$

$$\varphi_{n+1,pk+s}(x) = \frac{1}{p}\left(\varphi_{n,k}(x) + \frac{1-\alpha}{\alpha N} \sum_{\nu=0}^{N-1} \varepsilon_p^{\gamma(\nu)-\gamma(k)} \varphi_{n,\nu}(x) \right) + \frac{1}{p} \sum_{j=1}^{p-1} \varepsilon_p^{-js} \psi_{n,k}^{(j)}(x), \tag{2.12}$$

where $1 \le k \le N-1$, $0 \le s \le p-1$. Note also, that $w_{N-1}(x)$ can be expressed as

$$w_{N-1}(x) = \frac{1}{\alpha} \sum_{s=0}^{N-1} \varepsilon_p^{\gamma(s)} \varphi_{n,s}(x) = \sum_{k=0}^{N-1} \sum_{s=0}^{p-1} \gamma_{n+1,pk+s}\, \varphi_{n+1,pk+s}(x), \tag{2.13}$$

where $\gamma_{n+1,pk+s} := w_{N-1,pk+s}^{(n+1)}$.

For any functions $f_n \in V_n$ and $g_n \in W_n$ we write

$$f_n(x) = \sum_{k=0}^{N-1} C_{n,k}\, \varphi_{n,k}(x), \quad g_n(x) = \sum_{j=0}^{p-1} g_n^{(j)}(x), \tag{2.14}$$

where

$$g_n^{(j)}(x) = \sum_{k=0}^{N-1} D_{n,k}^{(j)} \psi_{n,k}(x),$$

and the coefficient sequences

$$\mathbf{C}_n = \{C_{n,k}\}, \quad \mathbf{D}_n^{(j)} = \{D_{n,k}^{(j)}\}, \quad 1 \le j \le p-1, \tag{2.15}$$

uniquely determine f_n and g_n, respectively. Let us describe the algorithms, in terms of the coefficient sequences (2.15), for decomposing $f_{n+1} \in V_{n+1}$ as the orthogonal sum of $f_n \in V_n$ and $g_n^{(j)} \in W_n^{(j)}$, and for reconstructing f_{n+1} from f_n and $g_n^{(j)}$.

As a consequence of (2.12) we observe that

$$\varphi_{n+1,pk+s}(x) = \sum_{\nu=0}^{N-1} A_{pk+s,\nu}^{(n)}\, \varphi_{n,\nu}(x) + \sum_{j=1}^{p-1} B_{pk+s,j}^{(n)} \psi_{n,k}^{(j)}(x), \tag{2.16}$$

where

$$A_{pk+s,\nu}^{(n)} = \begin{cases} 1/p + (1-\alpha)/(\alpha p N), & \nu = k, \\ \varepsilon_p^{\gamma(\nu)-\gamma(k)}(1-\alpha)/(\alpha p N), & \nu \ne k \end{cases} \quad \text{and} \quad B_{pk+s,j}^{(n)} = p^{-1} \varepsilon_p^{-js}.$$

Since $f_n + g_n = f_{n+1}$, it follows from (2.14) and (2.16) that

$$\sum_{\nu=0}^{N-1} C_{n,\nu}\, \varphi_{n,\nu}(x) + \sum_{j=1}^{p-1}\sum_{\nu=0}^{N-1} D_{n,\nu}^{(j)} \psi_{n,\nu}^{(j)}(x)$$

$$= \sum_{s=0}^{p-1}\sum_{k=0}^{N-1} C_{n+1,pk+s}\, \varphi_{n+1,pk+s}(x)$$

$$= \sum_{s,k} C_{n+1,pk+s}\left\{ \sum_{\nu=0}^{N-1} A_{pk+s,\nu}^{(n)}\, \varphi_{n,\nu}(x) + \sum_{j=1}^{p-1} B_{pk+s,j}^{(n)} \psi_{n,k}^{(j)}(x) \right\}$$

$$= \sum_{\nu}\left\{ \sum_{s,k} C_{n+1,pk+s} A_{pk+s,\nu}^{(n)} \right\} \varphi_{n,\nu}(x) + \sum_{j=1}^{p-1}\left\{ \sum_{s,k} C_{n+1,pk+s} B_{pk+s,j}^{(n)} \right\} \psi_{n,k}^{(j)}(x).$$

This implies that

$$C_{n,\nu} = \sum_{s,k} A_{pk+s,\nu}^{(n)} C_{n+1,pk+s}, \quad D_{n,\nu}^{(j)} = \sum_{s=0}^{p-1} B_{p\nu+s,j}^{(n)} C_{n+1,p\nu+s}. \tag{2.17}$$

Now, using (2.11) and (2.13), we obtain

$$\varphi_{n,\nu}(x) = \sum_{k=0}^{N-1}\sum_{s=0}^{p-1} Q_{pk+s,\nu}^{(n)}\, \varphi_{n+1,pk+s}(x),$$

where

$$Q_{pk+s,\nu}^{(n)} = \begin{cases} 1 - \varepsilon_p^{\gamma(k)}(1-\alpha)\gamma_{n+1,pk+s}/N, & k = \nu, \\ -\varepsilon_p^{\gamma(k)}(1-\alpha)\gamma_{n+1,pk+s}/N, & k \neq \nu. \end{cases}$$

Therefore, we have

$$\sum_{k,s} C_{n+1,pk+s}\, \varphi_{n+1,pk+s}(x)$$

$$= \sum_{\nu} C_{n,\nu}\left\{ \sum_{k,s} Q_{pk+s,\nu}^{(n)}\, \varphi_{n+1,pk+s}(x) \right\} + \sum_{j=1}^{p-1}\sum_{k=0}^{N-1} D_{n,k}^{(j)}\left\{ \sum_{s=0}^{p-1} \varepsilon_p^{js}\, \varphi_{n+1,pk+s}(x) \right\}$$

$$= \sum_{k,s}\left\{ \sum_{\nu} Q_{pk+s,\nu}^{(n)} C_{n,\nu} + \sum_{j} \varepsilon_p^{js} D_{n,k}^{(j)} \right\} \varphi_{n+1,pk+s}(x)$$

and so

$$C_{n+1,pk+s} = \sum_{\nu} Q_{pk+s,\nu}^{(n)} C_{n,\nu} + \sum_{j} \varepsilon_p^{js} D_{n,k}^{(j)}. \tag{2.18}$$

We remark that the decomposition and reconstruction algorithms based on formulas (2.17) and (2.18) have more simply structure than the similar algorithms constructed in [5] for the case of trigonometric wavelets.

To conclude this section, let us consider the case where $p = 2$, $N = 2^n$, and

$$b_k = \begin{cases} a_k, & 0 \le k \le N-1, \\ a_{N-k}, & N \le k \le 2N-1; \end{cases} \tag{2.19}$$

with any $a_k \neq 0$. Then, for all $k \in \{0, 1, \dots, N-1\}$,

$$\varphi_{n,k}(x) = \varphi_{n+1,2k}(x) + \varphi_{n+1,2k+1}(x), \quad \psi_{n,k}(x) = \varphi_{n+1,2k}(x) - \varphi_{n+1,2k+1}(x),$$

and thus

$$\varphi_{n+1,2k}(x) = \frac{1}{2}[\varphi_{n,k}(x) + \psi_{n,k}(x)], \quad \varphi_{n+1,2k+1}(x) = \frac{1}{2}[\varphi_{n,k}(x) - \psi_{n,k}(x)].$$

Hence, under the condition (2.19), instead of (2.17) and (2.18) we obtain the classical Haar discrete transforms.

3. **Periodic discrete p-adic wavelets**

Let us denote by $\langle k \rangle_p$ the remainder from the division of the integer k by the natural number p, and let $[a]$ be the integer part of a number a. For any $a \in \mathbb{R}_+$, the digits of the p-adic expansion

$$a = \sum_{\nu=1}^{\infty} a_{-\nu} p^{\nu-1} + \sum_{\nu=1}^{\infty} a_\nu p^{-\nu} \tag{3.1}$$

are defined by $a_{-\nu} = \langle [p^{1-\nu} a] \rangle_p$, $a_\nu = \langle [p^\nu a] \rangle_p$ (so, the finite representation for a p-adic rational a is taken). We can easily see that, for each $a \in \mathbb{R}_+$ there exists a natural number μ such that $a_{-\nu} = 0$ for all $\nu > \mu$ as well as that the first sum in (3.1) is equal to $[a]$. The representation (3.1) induces the operation of addition modulo p (or p-adic addition) on $\mathbb{R}_+$ as follows

$$a \oplus_p b := \sum_{\nu=1}^{\infty} \langle a_{-\nu} + b_{-\nu} \rangle_p p^{\nu-1} + \sum_{\nu=1}^{\infty} \langle a_\nu + b_\nu \rangle_p p^{-\nu}, \quad a, b \in \mathbb{R}_+.$$

As usual, the equality $c = a \ominus_p b$ means that $c \oplus_p b = a$.

For $N = p^n$, we set $\mathbb{Z}_N = \{0, 1, \dots, N-1\}$. Suppose that the space $\mathbb{C}_N$ consists of complex sequences $x = (\dots, x(-1), x(0), x(1), x(2), \dots)$, such that $x(j+N) = x(j)$ for all $j \in \mathbb{Z}$. An arbitrary sequence x from $\mathbb{C}_N$ is given if the values of $x(j)$ are given for $j \in \mathbb{Z}_N$; therefore, the element x is often identified with the vector $(x(0), x(1), \dots, x(N-1))$. The space $\mathbb{C}_N$ is equipped with the following natural inner product:

$$\langle x, y \rangle := \sum_{j=0}^{N-1} x(j) \overline{y(j)}.$$

For an arbitrary $j \in \mathbb{Z}_N$, let j^* denote the nonnegative integer defined by the condition $j \oplus_p j^* = 0$. For $p = 2$, we have $j^* = j$, and, for $p > 2$, the number j^* is p-adic opposite to j. For each $x \in \mathbb{C}_N$ we denote by $\widetilde{x}$ the vector from $\mathbb{C}_N$ such that

$\widetilde{x}(j)=\overline{x(j^*)}$ for all $j\in\mathbb{Z}_N$. Further, for $k,j\in\ \mathbb{Z}_N$, we set $\{k,j\}_p:=\sum\limits_{\nu=1}^{n}k_{\nu-n-1}j_{-\nu}$, where

$$k=\sum_{\nu=1}^{n}k_{-\nu}p^{\nu-1},\ \ j=\sum_{\nu=1}^{n}j_{-\nu}p^{\nu-1},\quad k_{-\nu},j_{-\nu}\in\{0,1,\dots,p-1\}.$$

The Vilenkin-Chrestenson functions $w_0^{(N)},w_1^{(N)},\dots,w_{N-1}^{(N)}$ for the space $\mathbb{C}_N$ are defined by the equalities $w_k^{(N)}(j)=\varepsilon_p^{\{k,j\}_p}$ and $w_k^{(N)}(l)=w_k^{(N)}(l+N)$, where $k,j\in\mathbb{Z}_N$, $l\in\mathbb{Z}$. For $n\geq 2$ and $p=2$, the Vilenkin-Chrestenson functions coincide with the Walsh functions and, in the case $n=1$ and $p\geq 2$, they are exponential functions: $w_k^{(p)}(j)=\varepsilon_p^{kj}$, $k,j\in\{0,1,\dots,p-1\}$.

The functions $w_0^{(N)},w_1^{(N)},\dots,w_{N-1}^{(N)}$ constitute an orthogonal basis in $\mathbb{C}_N$ and $\|w_k^{(N)}\|^2=N$ for all $k\in\mathbb{Z}_N$. To an arbitrary vector x from $\mathbb{C}_N$ the *Vilenkin-Chrestenson transform* assigns the sequence $\widehat{x}$ of the Fourier coefficients of x in the system $w_0^{(N)},w_1^{(N)},\dots,w_{N-1}^{(N)}$:

$$\widehat{x}(k):=\frac{1}{N}\sum_{j=0}^{N-1}x(j)\overline{w_k^{(N)}(j)},\quad k\in\mathbb{Z}_N.$$

For all $x,y\in\mathbb{C}_N$, we define the p-convolution $x*y$ by the formula

$$(x*y)(k):=\sum_{j=0}^{N-1}x(k\ominus_p j)y(j),\quad k\in\mathbb{Z}_N.$$

By a *unit N-periodic impulse* we mean the vector δ_N from $\mathbb{C}_N$ defined by the equality

$$\delta_N(j):=\begin{cases}1, & \text{if } j \text{ is divisible by } N,\\ 0, & \text{if } j \text{ is not divisible by } N.\end{cases}$$

The system of shifts $\{\delta_N(\cdot\ominus_p k)|\ k\in\mathbb{Z}_N\}$ is an orthonormal basis in $\mathbb{C}_N$ and

$$x(j)=(x*\delta_N)(j)=\sum_{k=0}^{N-1}x(k)\delta_N(j\ominus_p k),\quad j\in\mathbb{Z}_N,$$

for all $x\in\mathbb{C}_N$. For each $k\in\mathbb{Z}_N$ the p-adic shift operator $T_k:\ \mathbb{C}_N\to\mathbb{C}_N$ is defined as

$$(T_kx)(j):=x(j\ominus_p k),\quad x\in\mathbb{C}_N,\ j\in\mathbb{Z}_N.$$

It follows from the definitions that, for all $x,y\in\mathbb{C}_N$, the following relations hold:

$$\langle x,y\rangle=N\langle\widehat{x},\widehat{y}\rangle,\quad \widehat{x*y}=N\widehat{x}\widehat{y},\qquad \widehat{(T_kx)}(l)=\overline{w_k^{(N)}(l)}\widehat{x}(l),$$

$$\langle y,T_kx\rangle=y*\widetilde{x}(k),\quad \langle T_kx,T_ly\rangle=\langle x,T_{l\ominus_p k}y\rangle,\quad k,l\in\mathbb{Z}_N.$$

For $\nu=0,1,\dots,n$, we set $N_\nu=N/p^\nu$ and $\Delta_\nu=p^{\nu-1}$. The operators $D:\mathbb{C}_N\to\mathbb{C}_{N_1}$ and $U:\mathbb{C}_{N_1}\to\mathbb{C}_N$ given by the formulas

$$(Dx)(j):=x(pj),\quad j=0,1,\dots,N_1-1,$$

and

$$(Uy)(j) := \begin{cases} y(j/p) & \text{if } j \text{ is divisible by } p, \\ 0 & \text{if } j \text{ is not divisible by } p, \end{cases}$$

where $x \in \mathbb{C}_N$ and $y \in \mathbb{C}_{N_1}$ are called the *thickening sampling operator* and the *thinning sampling operator*, respectively. Note that $D(Uy) = y$ for all $y \in \mathbb{C}_{N_1}$. Further, suppose that $D^1 = D$, $U^1 = U$ and, for $v = 2, \dots, n$, we define the operators $D^v : \mathbb{C}_N \to \mathbb{C}_{N_v}$ and $U^v : \mathbb{C}_{N_v} \to \mathbb{C}_N$ by the formulas

$$(D^v x)(j) := x(p^v j), \qquad (U^v y)(j) := \begin{cases} y(j/p^v) & \text{if } j \text{ is divisible by } p^v, \\ 0 & \text{if } j \text{ is not divisible by } p^v, \end{cases}$$

where $x \in \mathbb{C}_N$ and $y \in \mathbb{C}_{N_v}$. For any $y \in \mathbb{C}_{N_v}$, the following relation holds: $\widehat{U^v y}(l) = p^{-v}\widehat{y}(l)$, $l \in \mathbb{Z}_N$, where, on the left-hand side, the Vilenkin-Chrestenson transform is taken in $\mathbb{C}_N$, while, on the righthand side, it is taken in $\mathbb{C}_{N_v}$.

Following the approach from [21, Chapter 3], we give the following definition.

Definition 3.1. Suppose that $u_0, u_1, \dots, u_{p-1} \in \mathbb{C}_N$. If the system

$$B(u_0, u_1, \dots, u_{p-1}) = \{T_{pk}u_0\}_{k=0}^{N_1-1} \cup \{T_{pk}u_1\}_{k=0}^{N_1-1} \cup \cdots \cup \{T_{pk}u_{p-1}\}_{k=0}^{N_1-1}$$

is an orthonormal basis in $\mathbb{C}_N$, then $B(u_0, u_1, \dots, u_{p-1})$ is called the *wavelet basis of the first stage in $\mathbb{C}_N$ generated by the collection of vectors $u_0, u_1, \dots, u_{p-1}$*.

The following theorem characterizes all the collections of vectors generating wavelet bases of the first stage in $\mathbb{C}_N$.

Theorem 3.1. *The collection of vectors $u_0, u_1, \dots, u_{p-1}$ generates a wavelet basis of the first stage in $\mathbb{C}_N$ if and only if the matrix*

$$A(l) := \frac{N}{\sqrt{p}} \begin{pmatrix} \widehat{u}_0(l) & \widehat{u}_1(l) & \dots & \widehat{u}_{p-1}(l) \\ \widehat{u}_0(l+N_1) & \widehat{u}_1(l+N_1) & \dots & \widehat{u}_{p-1}(l+N_1) \\ \widehat{u}_0(l+2N_1) & \widehat{u}_1(l+2N_1) & \dots & \widehat{u}_{p-1}(l+2N_1) \\ \vdots & \vdots & \dots & \vdots \\ \widehat{u}_0(l+(p-1)N_1) & \widehat{u}_1(l+(p-1)N_1) & \dots & \widehat{u}_{p-1}(l+(p-1)N_1) \end{pmatrix}$$

is unitary for $l = 0, 1, \dots, N_1 - 1$.

For each $1 \le m \le n$ we define the following *procedure for the construction of a wavelet basis of the first stage in $\mathbb{C}_N$*.

Step 1. Choose complex numbers b_l, $0 \le l \le p^m - 1$, satisfying the condition

$$\sum_{k=0}^{p-1} |b_{l+kp^{m-1}}|^2 = 1, \quad l = 0, 1, \dots, p^{m-1} - 1. \tag{3.2}$$

Step 2. Calculate $a_0, \dots, a_{p^m-1}$ by the formulas

$$a_j = p^{-m+1/2} \sum_{l=0}^{p^m-1} b_l \overline{w_l^{(p^m)}(j)}, \quad j = 0, 1, \dots, p^m - 1.$$

Step 3. Define a vector $u_0 \in \mathbb{C}_N$, for which

$$u_0(j) = \begin{cases} a_j, & 0 \le j \le p^m - 1, \\ 0, & p^m \le j \le p^n - 1. \end{cases} \tag{3.3}$$

Step 4. Find vectors $u_1, \dots, u_{p-1} \in \mathbb{C}_N$ such that, for all $l = 0, 1, \dots, N_1 - 1$, the matrix $A(l)$ is unitary.

Using Theorem 3.1, we can verify that the resulting collection of vectors $u_0, u_1, \dots, u_{p-1}$ generates a wavelet basis of the first stage in $\mathbb{C}_N$. In the case $p = 2$, step 4 of this procedure is carried out by the formula

$$u_1(j) = (-1)^j \overline{u_0(1 \oplus_2 j)}, \quad j \in \mathbb{Z}_N, \tag{3.4}$$

for $p > 2$, algorithms for the realization of this step were given in [28, Section 2.6] (see also [14, Section 2]). One of these algorithms is based on the Hausholder transform and can be described by the formulas

$$\widehat{u}_k(l) = \overline{\widehat{u}_0(l + kN_1)} \frac{1 - \widehat{u}_0(l)}{1 - \overline{\widehat{u}_0(l)}}, \tag{3.5}$$

$$\widehat{u}_k(l + jN_1) = \delta_{kj} - \frac{\widehat{u}_0(l + jN_1)\overline{\widehat{u}_0(l + kN_1)}}{1 - \overline{\widehat{u}_0(l)}}, \tag{3.6}$$

where δ_{kj} is the Kronecker delta, $k, j = 1, 2, \dots, p - 1$ and $l = 0, 1, \dots, N_1 - 1$.

Example 3.1. Suppose that $N > p$. Take $m = 1$ and $b_0 = 1$, $b_1 = \dots = b_{p-1} = 0$. Then the system $B(u_0, u_1, \dots, u_{p-1})$ is generated by the vectors

$$u_\mu = p^{-1/2}(1, \varepsilon_p^\mu, \varepsilon_p^{2\mu}, \dots, \varepsilon_p^{(p-1)\mu}, 0, 0, \dots, 0), \quad \mu = 0, 1, \dots, p - 1.$$

In particular, for $p = 2$, we have the *Haar basis of the first stage in* $\mathbb{C}_N$:

$$u_0 = (1/\sqrt{2}, 1/\sqrt{2}, 0, 0, \dots, 0), \quad u_1 = (1/\sqrt{2}, -1/\sqrt{2}, 0, 0, \dots, 0).$$

The following example is obtained by modifying the orthogonal wavelets constructed for the Cantor group in [24]; it corresponds to the case $m = p = 2$, $b_0 = 1$, $b_1 = a$, $b_2 = 0$, $b_3 = b$ in the procedure described above.

Example 3.2. Suppose that a and b are complex numbers such that $|a|^2 + |b|^2 = 1$. Suppose that $p = 2$ and $N \ge 4$, , and the vectors $u_0, u_1 \in \mathbb{C}_N$ are given by the equalities

$$u_0(0) = \frac{1+a+b}{2\sqrt{2}}, \quad u_0(1) = \frac{1+a-b}{2\sqrt{2}}, \quad u_0(2) = \frac{1-a-b}{2\sqrt{2}}, \quad u_0(3) = \frac{1-a+b}{2\sqrt{2}},$$

$$u_1(0) = \frac{1+a-b}{2\sqrt{2}}, \quad u_1(1) = -\frac{1+a+b}{2\sqrt{2}}, \quad u_1(2) = \frac{1-a+b}{2\sqrt{2}}, \quad u_1(3) = -\frac{1-a-b}{2\sqrt{2}},$$

under the condition that $u_0(j) = u_1(j) = 0$ for $4 \le j \le N - 1$. Then the vectors u_0, u_1 generate a wavelet basis of the first stage in $\mathbb{C}_N$. Note that, for $a = 1$, $b = 0$, the resulting wavelet basis $B(u_0, u_1)$ coincides with the Haar wavelet basis of the first stage described in Example 3.1.

The following two examples are similar to Examples 3 and 4 in [8].

Example 3.3. Suppose that $p=2$, $n>3$, and $m=3$. We set

$$(b_0, b_1, \dots, b_7) = \frac{1}{2}(1, a, b, c, 0, \alpha, \beta, \gamma),$$

where $|a|^2+|\alpha|^2=|b|^2+|\beta|^2=|c|^2+|\gamma|^2=1$. Then, by relation (3.3), we have

$$u_0(0) = \frac{1}{4\sqrt{2}}(1+a+b+c+\alpha+\beta+\gamma),$$

$$u_0(1) = \frac{1}{4\sqrt{2}}(1+a+b+c-\alpha-\beta-\gamma),$$

$$u_0(2) = \frac{1}{4\sqrt{2}}(1+a-b-c+\alpha-\beta-\gamma),$$

$$u_0(3) = \frac{1}{4\sqrt{2}}(1+a-b-c-\alpha+\beta+\gamma),$$

$$u_0(4) = \frac{1}{4\sqrt{2}}(1-a+b-c-\alpha+\beta-\gamma),$$

$$u_0(5) = \frac{1}{4\sqrt{2}}(1-a+b-c+\alpha-\beta+\gamma),$$

$$u_0(6) = \frac{1}{4\sqrt{2}}(1-a-b+c-\alpha-\beta+\gamma),$$

$$u_0(7) = \frac{1}{4\sqrt{2}}(1-a-b+c+\alpha+\beta-\gamma).$$

Further, we set $u_1(j)=u_0(j)=0$ for $8 \le j \le 2^n-1$, and we choose the other components of the vector u_1 so that relations (3.4) are valid, i.e.,

$$u_1(0)=\overline{u_0(1)}, \quad u_1(1)=-\overline{u_0(0)}, \quad u_1(2)=\overline{u_0(3)}, \quad u_1(3)=-\overline{u_0(2)},$$
$$u_1(4)=\overline{u_0(5)}, \quad u_1(5)=-\overline{u_0(4)}, \quad u_1(6)=\overline{u_0(7)}, \quad u_1(7)=-\overline{u_0(6)}.$$

The resulting pair u_0, u_1 generates a wavelet basis of the first stage in $\mathbb{C}_N$.

Example 3.4. Suppose that $p=3$, $n>2$, $m=2$ and

$$(b_0, b_1, \dots, b_8) = \frac{1}{\sqrt{3}}(1, a, \alpha, 0, b, \beta, 0, c, \gamma),$$

where $|a|^2+|b|^2+|c|^2=|\alpha|^2+|\beta|^2+|\gamma|^2=1$. Then, using (3.2) and (3.3), we obtain

$$u_0(0) = \frac{1}{3\sqrt{3}}(1+a+b+c+\alpha+\beta+\gamma),$$

$$u_0(1) = \frac{1}{3\sqrt{3}}(1+a+\alpha+(b+\beta)\varepsilon_3^2+(c+\gamma)\varepsilon_3),$$

$$u_0(2) = \frac{1}{3\sqrt{3}}(1 + a + \alpha + (b + \beta)\varepsilon_3 + (c + \gamma)\varepsilon_3^2),$$

$$u_0(3) = \frac{1}{3\sqrt{3}}(1 + (a + b + c)\varepsilon_3^2 + (\alpha + \beta + \gamma)\varepsilon_3),$$

$$u_0(4) = \frac{1}{3\sqrt{3}}(1 + c + \beta + (a + \gamma)\varepsilon_3^2 + (b + \alpha)\varepsilon_3),$$

$$u_0(5) = \frac{1}{3\sqrt{3}}(1 + b + \gamma + (a + \beta)\varepsilon_3^2 + (c + \alpha)\varepsilon_3),$$

$$u_0(6) = \frac{1}{3\sqrt{3}}(1 + (a + b + c)\varepsilon_3 + (\alpha + \beta + \gamma)\varepsilon_3^2),$$

$$u_0(7) = \frac{1}{3\sqrt{3}}(1 + b + \gamma + (a + \beta)\varepsilon_3 + (c + \alpha)\varepsilon_3^2),$$

$$u_0(8) = \frac{1}{3\sqrt{3}}(1 + c + \beta + (a + \gamma)\varepsilon_3 + (b + \alpha)\varepsilon_3^2),$$

where $\varepsilon_3 = \exp(2\pi i/3)$. We set $u_0(j) = u_1(j) = u_2(j) = 0$ for $9 \le j \le 3^n - 1$ and use (3.5) to define the other components of the vectors $u_1, u_2 \in \mathbb{C}_N$ so that the matrix

$$\frac{9}{\sqrt{3}} \begin{pmatrix} \widehat{u}_0(l) & \widehat{u}_1(l) & \widehat{u}_2(l) \\ \widehat{u}_0(l+3) & \widehat{u}_1(l+3) & \widehat{u}_2(l+3) \\ \widehat{u}_0(l+6) & \widehat{u}_1(l+6) & \widehat{u}_2(l+6) \end{pmatrix}$$

is unitary for $l = 0, 1, 2$. The resulting collection of the vectors u_0, u_1, u_2 generates a wavelet basis of the first stage in $\mathbb{C}_N$.

The values of the parameters b_l in Examples 3.2-3.4 are universal in the sense that they occur not only in the construction of wavelet bases in $\mathbb{C}_N$, but also in the corresponding examples for the spaces $\ell^2(\mathbb{Z}_+)$ and $L^2(\mathbb{R}_+)$. At the same time, the construction of orthogonal wavelets on the Cantor and Vilenkin groups (as well as on the half-line $\mathbb{R}_+$; see [8], [10]) requires some additional constraint related to the requirement that the masks have no blocking sets (so, in Example 2, the pair $a = 0$, $b = 1$ leads to a wavelet basis in the space $\mathbb{C}_N$, while, in the original example due to Lang, this pair corresponds to a linearly dependent system; see also Example 2 in [8]). The great freedom of choice of the values of the parameters in the construction of orthogonal wavelets in the space $\mathbb{C}_N$ by the method described in this paper becomes apparent due to the fact that, according to step 1 of the procedure, for $(b_0, b_1, \dots, b_{p^m-1})$ we can choose *any complex vector of dimension* p^m *satisfying condition* (3.2) (compare with the construction of discrete Daubechies wavelets in [3] and [21]). This property is important for applications, because it extends the range of applications of the well-known adaptive signal-approximation methods (see, for example, Chapters 8-10 in Mallat's book [26]).

Definition 3.2. Suppose that $m \in \mathbb{N}$, $m \le n$. By a *sequence of orthogonal wavelet filters of the mth stage* we mean a sequence of vectors

$$u_0^{(1)}, u_1^{(1)}, \dots, u_{p-1}^{(1)}, \quad \dots, \quad u_0^{(m)}, u_1^{(m)}, \dots, u_{p-1}^{(m)},$$

such that $u_\mu^{(\nu)} \in \mathbb{C}_{N_{\nu-1}}$ for $\nu = 1, 2, \dots, m$, $\mu = 0, 1, \dots, p-1$ and the matrices

$$A^{(\nu)}(l) := \frac{N}{\sqrt{p}} \begin{pmatrix} \widehat{u}_0^{(\nu)}(l) & \dots & \widehat{u}_{p-1}^{(\nu)}(l) \\ \widehat{u}_0^{(\nu)}(l+N_\nu) & \dots & \widehat{u}_{p-1}^{(\nu)}(l+N_\nu) \\ \widehat{u}_0^{(\nu)}(l+2N_\nu) & \dots & \widehat{u}_{p-1}^{(\nu)}(l+2N_\nu) \\ \dots & \dots & \dots \\ \widehat{u}_0^{(\nu)}(l+(p-1)N_\nu) & \dots & \widehat{u}_{p-1}^{(\nu)}(l+(p-1)N_\nu) \end{pmatrix}$$

are unitary for $\nu = 1, 2, \dots, m$, $l = 0, 1, \dots, N_\nu - 1$.

Theorem 3.2. *Suppose that the collection of vectors $u_0, u_1, \dots, u_{p-1}$ generates a wavelet basis of the first stage in $\mathbb{C}_N$. For a given $m \in \mathbb{N}$, $m \le n$, set*

$$u_\mu^{(1)}(j) = u_\mu(j), \quad u_\mu^{(\nu)}(j) = \Delta_\nu^{-1} \sum_{k=0}^{\Delta_\nu - 1} u_\mu^{(1)}(j + kN_{\nu-1}), \quad j \in \mathbb{Z}_{N_{\nu-1}}, \tag{3.7}$$

where $\nu = 2, \dots, m$, $\mu = 0, 1, \dots, p-1$. Then the vectors

$$u_0^{(1)}, u_1^{(1)}, \dots, u_{p-1}^{(1)}, \quad \dots, \quad u_0^{(m)}, u_1^{(m)}, \dots, u_{p-1}^{(m)},$$

constitute a sequence of orthogonal wavelet filters of the mth stage.

Thus, from a given vector $u_0 \in \mathbb{C}_N$, defined by (3.2) and (3.3) we can, first, find a wavelet basis of the first stage $u_0, u_1, \dots, u_{p-1}$, using (3.4) or (3.5), and then, using (3.6) obtain the sequence of orthogonal wavelet filters of the *m*th stage. Denote by $\oplus$ the direct sum of the subspaces of the space $\mathbb{C}_N$. By the theorem that follows, from any sequence of orthogonal wavelet filters of the *m*th stage we can construct an orthonormal wavelet basis in $\mathbb{C}_N$.

Theorem 3.3. *Suppose that a sequence of orthogonal wavelet filters of the mth stage is given in the space $\mathbb{C}_N$:*

$$u_0^{(1)}, u_1^{(1)}, \dots, u_{p-1}^{(1)}, \quad \dots, \quad u_0^{(m)}, u_1^{(m)}, \dots, u_{p-1}^{(m)}.$$

Let $\varphi^{(1)} = u_0^{(1)}$, $\psi_\mu^{(1)} = u_\mu^{(1)}$, $\mu = 1, \dots, p-1$, and define $\varphi^{(\nu)}$, $\psi_\mu^{(\nu)}$ for $\nu = 2, \dots, m$, $\mu = 1, \dots, p-1$ by the formulas

$$\varphi^{(\nu)} = \varphi^{(\nu-1)} * U^{\nu-1} u_0^{(\nu)}, \quad \psi_\mu^{(\nu)} = \varphi^{(\nu-1)} * U^{\nu-1} u_\mu^{(\nu)}.$$

Further, for $\nu = 1, \dots, m$, $\mu = 1, \dots, p-1$, we set

$$\varphi_{-\nu,k} = T_{p^\nu k}\, \varphi^{(\nu)}, \quad \psi_{-\nu,k}^{(\mu)} = T_{p^\nu k} \psi_\mu^{(\nu)}, \quad k = 0, 1, \dots, N_\nu - 1,$$

and define the subspaces

$$V_{-\nu} = \operatorname{span}\{\varphi_{-\nu,k}\}_{k=0}^{N_\nu-1}, \quad W_{-\nu}^{(\mu)} = \operatorname{span}\{\psi_{-\nu,k}^{(\mu)}\}_{k=0}^{N_\nu-1},$$

$$W_{-\nu} = W_{-\nu}^{(1)} \oplus \cdots \oplus W_{-\nu}^{(p-1)}.$$

Then the following expansion holds:

$$\mathbb{C}_N = W_{-1} \oplus W_{-2} \oplus \cdots \oplus W_{-m} \oplus V_{-m} \tag{3.8}$$

and, for each $\nu = 1, 2, \dots, m$ the following properties are valid:

(a) $V_{-\nu} = V_{-\nu-1} \oplus W_{-\nu-1}$;
(b) $\{\varphi_{-\nu,k}\}_{k=0}^{N_\nu-1}$ *is an orthonormal basis in* $V_{-\nu}$;
(c) $\{\psi_{-\nu,k}^{(1)}\}_{k=0}^{N_\nu-1} \cup \cdots \cup \{\psi_{-\nu,k}^{(p-1)}\}_{k=0}^{N_\nu-1}$ *is an orthonormal basis in* $W_{-\nu}$.

This theorem justifies the method of constructing subspaces $V_{-1}, \dots, V_{-n}$ in $\mathbb{C}_N$ with the following properties:

(i) $V_{-\nu-1} \subset V_{-\nu}$ for all $\nu \in \{1, 2, \dots n\}$;
(ii) for each $\nu \in \{1, 2, \dots n\}$, there exists a vector $\varphi^{(\nu)} \in V_{-\nu}$ such that the system $\{T_{p^\nu k}\, \varphi^{(\nu)}\}_{k=0}^{N_\nu-1}$ is an orthonormal basis in $V_{-\nu}$;
(iii) for each $1 \le m \le n$, relation (3.7) is valid;
(iv) for each $\nu \in \{1, 2, \dots n\}$ there exist vectors $\psi_1^{(\nu)}, \dots, \psi_{p-1}^{(\nu)} \in W_{-\nu}$ such that the system $\bigcup_{\mu=1}^{p-1}\{T_{p^\nu k}\psi_\mu^{(\nu)}\}_{k=0}^{N_\nu-1}$ is an orthonormal basis in $W_{-\nu}$.

Theorems 3.1-3.3 are proved by the author in [16]. A similar construction in the space $L^2(\mathbb{R}^d)$ is well-known and is related to the notion of *multiresolution analysis*. According to the terminology used in the theory of multiresolution analysis, the sequence $\{\varphi^{(\nu)}\}_{\nu=1}^n$ in property (ii) it is natural to call a *scaling sequence* in $\mathbb{C}_N$.

In particular, for $p = 2$, $n = 3$, using Theorem 3.3, we obtain three orthonormal wavelet bases in $\mathbb{C}_8$:

$$\{\psi_{-1,k}\}_{k=0}^3 \cup \{\varphi_{-1,k}\}_{k=0}^3 \qquad (m = 1),$$

$$\{\psi_{-1,k}\}_{k=0}^3 \cup \{\psi_{-2,k}\}_{k=0}^1 \cup \{\varphi_{-2,k}\}_{k=0}^1 \qquad (m = 2),$$

$$\{\psi_{-1,k}\}_{k=0}^3 \cup \{\psi_{-2,k}\}_{k=0}^1 \cup \{\psi_{-3,0}\} \cup \{\varphi_{-3,0}\} \qquad (m = 3).$$

In the Haar case (see Example 3.1), these bases consist of the vectors

$$\varphi_{-1,0} = \frac{1}{\sqrt{2}}(1,1,0,0,0,0,0,0), \quad \psi_{-1,0} = \frac{1}{\sqrt{2}}(1,-1,0,0,0,0,0,0),$$

$$\varphi_{-1,1} = \frac{1}{\sqrt{2}}(0,0,1,1,0,0,0,0), \quad \psi_{-1,1} = \frac{1}{\sqrt{2}}(0,0,1,-1,0,0,0,0),$$

$$\varphi_{-1,2} = \frac{1}{\sqrt{2}}(0,0,0,0,1,1,0,0), \quad \psi_{-1,2} = \frac{1}{\sqrt{2}}(0,0,0,0,1,-1,0,0),$$

$$\varphi_{-1,3} = \frac{1}{\sqrt{2}}(0,0,0,0,0,0,1,1), \qquad \psi_{-1,3} = \frac{1}{\sqrt{2}}(0,0,0,0,0,0,1,-1),$$

$$\varphi_{-2,0} = \frac{1}{2}(1,1,1,1,0,0,0,0), \qquad \psi_{-2,0} = \frac{1}{2}(1,1,-1,-1,0,0,0,0),$$

$$\varphi_{-2,1} = \frac{1}{2}(0,0,0,0,1,1,1,1), \qquad \psi_{-2,1} = \frac{1}{2}(0,0,0,0,1,1,-1,-1),$$

$$\varphi_{-3,0} = \frac{1}{2\sqrt{2}}(1,1,1,1,1,1,1,1), \quad \psi_{-3,0} = \frac{1}{2\sqrt{2}}(1,1,1,1,-1,-1,-1,-1).$$

In the general case, the orthogonal projections $P_{-\nu} : \mathbb{C}_N \to V_{-\nu}$ and $Q_{-\nu} : \mathbb{C}_N \to W_{-\nu}$ act by the formulas

$$P_{-\nu}x = \sum_{k=0}^{N_\nu-1} \langle x, \varphi_{-\nu,k}\rangle \varphi_{-\nu,k}, \quad Q_{-\nu}x = \sum_{\mu=1}^{p-1}\sum_{k=0}^{N_\nu-1} \langle x, \psi^{(\mu)}_{-\nu,k}\rangle \psi^{(\mu)}_{-\nu,k}. \tag{3.9}$$

Suppose that I is the identity operator on $\mathbb{C}_N$. Setting $P_0 = I$, $V_0 = \mathbb{C}_N$ and using Theorem 3.3 for any $x \in \mathbb{C}_N$, we obtain the equalities

$$x = P_{-\nu}x + \sum_{k=1}^{\nu} Q_{-k}x, \;\; P_{-\nu+1}x = P_{-\nu}x + Q_{-\nu}x, \quad \nu = 1,2,\dots,n.$$

An arbitrary vector x from $\mathbb{C}_N$ can be regarded as the input signal $a_0 = x$ and, for $\nu = 1,2,\dots,m$, we can set

$$a_\nu = D(a_{\nu-1} * \widetilde{u}^{(\nu)}_0), \;\; d^{(\mu)}_\nu = D(a_{\nu-1} * \widetilde{u}^{(\nu)}_\mu), \quad \mu = 1,\dots,p-1. \tag{3.10}$$

We can easily see that the components of the vectors a_ν and $d^{(\mu)}_\nu$ are the coefficients of the expansions (3.8) for a chosen x. The application of formulas (3.9) constitutes the *phase of the analysis* of the signal x and yields the collection of vectors

$$d^{(1)}_1,\dots,d^{(1)}_{p-1},\dots,d^{(m)}_1,\dots,d^{(m)}_{p-1}, a_m. \tag{3.11}$$

The inverse passage from the collection (3.10) to the original vector x constitutes the *reconstruction phase* and is defined by the formulas

$$a_{\nu-1} = u^{(\nu)}_0 * Ua_\nu + \sum_{\mu=1}^{p-1} u^{(\nu)}_\mu * Ud^{(\nu)}_\mu, \quad \nu = m, m-1,\dots,1. \tag{3.12}$$

Formulas (3.9) and (3.11) specify the *direct and inverse discrete wavelet transforms* associated with the sequence of wavelet filters $u^{(1)}_0, u^{(1)}_1,\dots,u^{(1)}_{p-1}, \dots, u^{(m)}_0, u^{(m)}_1,\dots,u^{(m)}_{p-1}$, and are realized by using fast algorithms (cf. [21, Section 3.2], [28, Section 4]).

Remark 3.1. Suppose that $m \in \mathbb{N}$, $m \le n$. For a given sequence of vectors

$$u^{(1)}_0,\dots,u^{(1)}_{p-1},\; v^{(1)}_0,\dots,v^{(1)}_{p-1}, \quad \dots, \quad u^{(m)}_0,\dots,u^{(m)}_{p-1},\; v^{(m)}_0,\dots,v^{(m)}_{p-1}, \tag{3.13}$$

such that $u_\mu^{(\nu)}, v_\mu^{(\nu)} \in \mathbb{C}_{N_{\nu-1}}$ for $\nu = 1, 2, \dots, m$, $\mu = 0, 1, \dots, p-1$, we introduce the matrices $A^{(\nu)}(l)$ just as in Definition 3.2 and set

$$\overline{B}^{(\nu)}(l) := \frac{N}{\sqrt{p}} \begin{pmatrix} \overline{\widehat{v}_0^{(\nu)}(l)} & \dots & \overline{\widehat{v}_{p-1}^{(\nu)}(l)} \\ \overline{\widehat{v}_0^{(\nu)}(l+N_\nu)} & \dots & \overline{\widehat{v}_{p-1}^{(\nu)}(l+N_\nu)} \\ \overline{\widehat{v}_0^{(\nu)}(l+2N_\nu)} & \dots & \overline{\widehat{v}_{p-1}^{(\nu)}(l+2N_\nu)} \\ \dots & \dots & \dots \\ \overline{\widehat{v}_0^{(\nu)}(l+(p-1)N_\nu)} & \dots & \overline{\widehat{v}_{p-1}^{(\nu)}(l+(p-1)N_\nu)} \end{pmatrix}^T,$$

where T denotes transposition. We say that the vectors (3.12) constitute a *sequence of biorthogonal wavelet filters of the mth stage* if

$$\overline{B}^{(\nu)}(l) A^{(\nu)}(l) = E_p, \quad \nu = 1, 2, \dots, m;\ l = 0, 1, \dots, N_\nu - 1,$$

where E_p is the identity matrix of order p. Using this definition, we can generalize the construction given above to the biorthogonal case and, instead of Examples 3.2-3.4, obtain the discrete analogs of the corresponding examples from [12] and [14].

Remark 3.2. Suppose that $\{w_k\}_{k=0}^\infty$ is the generalized Walsh system determined from the given number $p \geq 2$ and generating an orthonormal basis in the L^2-space on the interval $\Delta = [0, 1)$ (the case $p = 2$ corresponds to the classical Walsh system; see, for example, [1]). To each sequence $x = (x_0, x_1, \dots)$ from $\ell^2(\mathbb{Z}_+)$ we assign the function $\widehat{x} := \sum_{k=0}^{\infty} x_k w_k$ in $L^2(\Delta)$. Using this mapping instead of the Vilenkin-Chrestenson transform, we can prove analogs of Theorems 3.1-3.3 for the space $\ell^2(\mathbb{Z}_+)$ (compare [21, Chapter 4]) and obtain the discrete nonperiodic analogs of the wavelet bases from [8] and [14].

Further discussions and possible applications of periodic wavelets considered in this paper can be found in the works [13] and [19].

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