Український математичний вісник
Том 9 **(2012)**, № 2, 175 – 190

Examples of frames on the Cantor dyadic group

Yuri A. Farkov

(*Presented by R. M. Trigub*)

Abstract. In this expository paper, we present two ways to constract frames on the locally compact Cantor dyadic group. The first approach gives a Parseval frame related to the generalized Walsh–Dirichlet kernel while the second approach includes the Daubechies type "admissible condition" and leads to dyadic compactly supported wavelet frames. The corresponding wavelet constructions on the Cantor and Vilenkin groups (as well as on the half-line $\mathbb{R}_+$) requires an additional constraint related to the requirement that the masks have no blocking sets.

2010 MSC. 42C10, 42C15, 42C40, 43A70.

Key words and phrases. Wavelets, frames, Walsh functions, Walsh–Dirichlet kernel, Cantor dyadic group, Vilenkin groups.

1. Introduction

The Walsh functions can be considered as characters of the Cantor dyadic group. This fact was first recognized by Gelfand in 1940s, who offered to Vilenkin study series with respect to characters of a large class of Abelian groups which includes the Cantor dyadic group as a special case (see [1, p. 3]). The basic properties of the Cantor dyadic group are given in Pontryagin's book [2]. Concerning its applications to the theory of trigonometric Fourier series see, e.g., [3]. At present the Walsh analysis is an actively developing domain of the harmonic analysis (see, e.g., [4–9]).

Let us recall definitions of the dyadic field $\mathbb{F}$ and the Cantor dyadic group G. Denote by $\mathbb{F}_2$ the field of order 2, with elements $\{0, 1\}$. Then the *dyadic field* $\mathbb{F}$ is the subset of $\prod_{j\in\mathbb{Z}}\mathbb{F}_2$ consisting of sequences

$$x = (x_j) = (\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots),$$

Received 9.09.2011

for which $x_j \to 0$ as $j \to -\infty$. Addition on $\mathbb{F}$ is the coordinate-wise addition modulo 2 :

$$(z_j) = (x_j) \oplus (y_j) \Longleftrightarrow z_j = x_j + y_j \pmod 2,$$

while multiplication on $\mathbb{F}$ follows the rule

$$(z_j) = (x_j) \cdot (y_j) \Longleftrightarrow z_j = \sum_{l+k=j} x_l y_k \pmod 2.$$

Define projections $\pi_j : \mathbb{F} \to \{0,1\}$ by $\pi_j(x) = x_j$ where $x = (x_j)$ and, for each $n \in \mathbb{Z}$, choose an unit element $e_n \in \mathbb{F}$ such that

$$\pi_j(e_n) = \begin{cases} 1, & j = n, \\ 0, & \text{otherwise}. \end{cases}$$

Notice that $e_n \cdot x = (x_{j-n})$ for all $x = (x_j) \in \mathbb{F}$.

Denote by θ the zero sequence in $\mathbb{F}$. We see that for each $x \in \mathbb{F}$ with $x \neq \theta$ there exists $s(x) \in \mathbb{Z}$ such that

$$x_{s(x)} = 1 \quad \text{and} \quad x_j = 0 \quad \text{for} \quad j < s(x).$$

There is a non-Archimedean norm on $\mathbb{F}$. Indeed, set $\|\theta\| = 0$ and for each $x \in \mathbb{F}$ with $x \neq \theta$ set $\|x\| = 2^{-s(x)}$. Then

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad \text{and} \quad \|x \cdot y\| = \|x\|\|y\|$$

for $x, y \in \mathbb{F}$. The *Cantor dyadic group* G can be defined as the additive group of the dyadic field $\mathbb{F}$, with the topology induced by $\|\cdot\|$.

The sets

$$U_l := \{(x_j) \in G \mid x_j = 0 \text{ for } j < l\}, \quad l \in \mathbb{Z},$$

form a complete system neighbourhoods of θ and they satisfy the following properties:

- $U_{l+1} \subset U_l$, $\bigcap U_l = \{\theta\}$, $\bigcup U_l = G$;
- each U_l is a compact open subgroup of G;
- each U_l is homeomorphic to the Cantor ternary set.

The group G is a locally compact abelian group. Denote by μ the Haar measure on G normalized so that $\mu(U) = 1$ where $U := U_0$.

One can show that G is self-dual. The duality pairing on G takes $x, \omega \in G$ to

$$(x, \omega) := (-1)^{\pi_{-1}(x \cdot \omega)}.$$

For any nonzero a, the multiplication-by-a map is an automorphism of G, with adjoint also multiplication-by-a. Let A be the automorphism of G which coincides with the multiplication-by-e_{-1} (i.e., $(Ax)_j = x_{j+1}$ for $x = (x_j) \in G$). Notice that A takes U to the larger subgroup U_{-1} and that $A^{-l}(U) = U_l$ for $l \in \mathbb{Z}$.

As usual, let $\mathbb{R}_+ = [0, +\infty)$. We define a map $\lambda : G \to \mathbb{R}_+$ by

$$\lambda(x) = \sum_{j \in \mathbb{Z}} x_j 2^{-j-1}, \qquad x = (x_j) \in G.$$

Take in G a discrete subgroup $H = \{(x_j) \in G \,|\, x_j = 0 \text{ for } j \geq 0\}$. The image of H under λ is the set of non-negative integers: $\lambda(H) = \mathbb{Z}_+$. For every $\alpha \in \mathbb{Z}_+$, let $h_{[\alpha]}$ denote the element of H such that $\lambda(h_{[\alpha]}) = \alpha$. Note that $h_{[1]} = e_{-1}$ since $\lambda(e_n) = 1/2^{n+1}$ for $n \in \mathbb{Z}$.

The *Walsh functions* on G can be defined by

$$W_\alpha(x) = (x, h_{[\alpha]}), \quad x \in G, \quad \alpha \in \mathbb{Z}_+.$$

It is well known that $\{W_\alpha\}$ is an orthonormal basis in $L^2(U)$ (e.g., [4, 6]).

We write $d\mu(x) = dx$. For any function $f \in L^1(G) \cap L^2(G)$ the Fourier transform $\widehat{f}$, defined by

$$\widehat{f}(\omega) = \int_G f(x)(x, \omega)\, dx, \quad \omega \in G,$$

belongs to $L^2(G)$. The Fourier operator

$$\mathcal{F} \,:\, L^1(G) \cap L^2(G) \to L^2(G), \quad \mathcal{F}f = \widehat{f},$$

extends uniquely to $L^2(G)$. By the Plancherel theorem,

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g}\, \rangle \quad \text{for all} \quad f, g \in L^2(G).$$

For any $\psi \in L^2(G)$ we let

$$\psi_{j,\alpha}(x) := 2^{j/2} \psi(A^j x \oplus h_{[\alpha]}), \quad j \in \mathbb{Z},\ \alpha \in \mathbb{Z}_+. \tag{1.1}$$

In the sequel, $\mathbf{1}_E$ stands for the characteristic function of a subset E of G.

Example 1.1. The Haar wavelet on G can be defined by

$$\psi_H(x) = \begin{cases} 1, & x \in U_1, \\ -1, & x \in U \setminus U_1, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, if $\psi = \psi_H$ then $\{\psi_{j,\alpha} : j \in \mathbb{Z},\ h \in H\}$ is the Haar basis for $L^2(G)$ (cf. [10, 11]). Further, we have

$$\psi_H(x) = \varphi_H(Ax) - \varphi_H(Ax \oplus h_{[1]}),$$

where $\varphi_H := \mathbf{1}_U$ is the Haar function for G. Also, the following two equalities are true:

$$\varphi_H(x) = \varphi_H(Ax) + \varphi_H(Ax \oplus h_{[1]}) \quad \text{and} \quad \widehat{\varphi}_H = \varphi_H.$$

The latter equality, in fact, shows that the Haar and Shannon wavelets coincide on G (see [12] for the details).

The following two propositions are well-known (e.g., [13]):

Proposition 1.1. *A sequence $\{g_m\}$ is a Parseval frame for a Hilbert space $\mathcal{H}$ if and only if the following formula holds for every $f \in \mathcal{H}$:*

$$f = \sum_{m \in M} \langle f, g_m \rangle g_m.$$

Proposition 1.2. *Let $\{g_m\}$ be a frame for $\mathcal{H}$ and let $\mathcal{P} : \mathcal{H} \to \mathcal{H}$ be an orthogonal projection. Then $\{\mathcal{P}g_m\}$ is a frame for $\mathcal{P}(\mathcal{H})$ with the same frame bounds. In particular, if $\{g_m\}$ is an orthonormal basis for $\mathcal{H}$, then $\{\mathcal{P}g_m\}$ is a Parseval frame for $\mathcal{P}(\mathcal{H})$.*

Example 1.2. If, for $x \in G$, we define $\varphi(x) = 2^{-1}\mathbf{1}_U(A^{-1}x)$, then

$$\varphi(x) = \varphi(Ax) + \varphi(Ax \oplus h_{[3]}) \quad \text{and} \quad \varphi(x) = \varphi(Ax \oplus h_{[1]}) + \varphi(Ax \oplus h_{[2]}).$$

According Theorem 3.4 below, each function

$$\psi(x) = \varphi(Ax) - \varphi(Ax \oplus h_{[3]}) \quad \text{and} \quad \psi(x) = -\varphi(Ax \oplus h_{[1]}) + \varphi(Ax \oplus h_{[2]}),$$

gives by (1.1) a Parseval frame for $L^2(G)$.

2. Frames related to the generalized Walsh–Dirichlet kernel

Let us recall that $\{\psi_{j,\alpha}\}$ is a *frame* for $L^2(G)$ if there exist positive constants c_0, c_1 such that, for every $f \in L^2(G)$,

$$c_0\|f\|^2 \le \sum_{j,\alpha} |\langle f, \psi_{j,\alpha}\rangle|^2 \le c_1\|f\|^2.$$

The constants c_0 and c_1 are known respectively as lower and upper frame bounds. If $c_0 = c_1$, we have a *tight frame*; in this case,

$$f = {c_0}^{-1}\sum_{j,\alpha}\langle f, \psi_{j,\alpha}\rangle\psi_{j,\alpha}, \qquad f \in L^2(G).$$

A frame $\{\psi_{j,\alpha}\}$ will be called a *Parseval frame* if $c_0 = c_1 = 1$.

Let γ be a non zero element in G. The function $\mathcal{D}_\gamma : G \to \mathbb{R}$ defined by

$$\mathcal{D}_\gamma(x) := \int_{\gamma U} (x, \omega)\, d\omega, \quad x \in G,$$

is called *the generalized Walsh–Dirichlet kernel* [4]. It is immediate from the definition that $\widehat{\mathcal{D}}_\gamma = \mathbf{1}_{\gamma U}$. In particular, if $\gamma = e_0$ then $\mathcal{D}_\gamma$ is the Haar function φ_H, and if $\gamma = e_1$ then we can write $\mathcal{D}_\gamma(x) = 2^{-1}\mathbf{1}_U(A^{-1}x)$ (cf. Example 1.2).

Define

$$V_j(\gamma) := \{f \in L^2(G) : \widehat{f}(\omega) = 0 \text{ for } \omega \in G \setminus A^j(\gamma U)\}, \qquad j \in \mathbb{Z},\ \gamma \in U_1 \setminus U_2. \quad (2.1)$$

It is clear that the subspaces $V_j(\gamma)$ satisfy the following conditions:

$$V_j(\gamma) \subset V_{j+1}(\gamma), \quad \overline{\bigcup V_j(\gamma)} = L^2(G) \quad \bigcap V_j(\gamma) = \{\theta\}$$

(compare with the definition of an MRA below).

Proposition 2.1. *Let $\varphi = \mathcal{D}_\gamma$ and $\psi = \mathcal{D}_{A\gamma} - \mathcal{D}_\gamma$ where $\gamma \in U_1 \setminus U_2$. Suppose that the subspaces $V_j(\gamma)$ are defined as in* (2.1)*. Then the following are true:*

(*a*) *$\{\varphi(\cdot \oplus h) : h \in H\}$ is a Parseval frame for $V_0(\gamma)$;*

(*b*) *$\{\varphi_{j,\alpha} : \alpha \in \mathbb{Z}_+\}$ is a Parseval frame for $V_j(\gamma)$;*

(c) if $f \in V_j(\gamma)$ *then*

$$f(x) = \sum_{h \in H} f(A^{-j}h)\varphi(A^j x \oplus h), \quad x \in G; \tag{2.2}$$

(d) $\{\psi_{j,\alpha} : j \in \mathbb{Z},\ \alpha \in \mathbb{Z}_+\}$ *is a Parseval frame for* $L^2(G)$.

Proof. For $\gamma = e_0$ we have $\varphi = \varphi_U$ and the subspaces $V_j(\gamma)$ form the Haar multiresolution analysis in $L^2(G)$. In this case, $\psi = \psi_H$ (see Example 1.1). Now, let $\gamma \neq e_0$ and assume that $E = \gamma U$. Then the linear mapping

$$\mathcal{P} : L^2(U) \to L^2(U), \quad \mathcal{P}f = f \cdot \mathbf{1}_E,$$

is an orthogonal projection. In fact, let $\mathcal{L}_0(E)$ be the closure of the linear span of $\{W_\alpha \cdot \mathbf{1}_E \mid \alpha \in \mathbb{Z}_+\}$ in $L^2(U)$. If $f \in L^2(U)$ and $g \in \mathcal{L}_0(E)$, then

$$\langle f, g\rangle = \int_U f(t)\overline{g(t)}\,dt = \int_E f(t)\overline{g(t)}\,dt = \int_U \mathcal{P}f(t)\overline{g(t)}\,dt = \langle \mathcal{P}f, g\rangle.$$

Hence,

$$\langle f - \mathcal{P}f, g\rangle = 0 \quad \text{for all } g \in \mathcal{L}_0(E).$$

Since $\{W_\alpha \mid \alpha \in \mathbb{Z}_+\}$ is an orthonormal basis for $L^2(U)$, by Proposition 1.2 we obtain that $\{W_\alpha \mathbf{1}_E \mid \alpha \in \mathbb{Z}_+\}$ is a Parseval frame for $\mathcal{L}_0(E)$. But for $\varphi_{0,\alpha}(\cdot) = \varphi(\cdot \oplus h_{[\alpha]})$ with $\varphi = \mathcal{D}_\gamma$ we have

$$\widehat{\varphi}_{0,\alpha}(\omega) = W_\alpha(\omega)\widehat{\varphi}(\omega) = W_\alpha(\omega)\mathbf{1}_E(\omega), \quad \alpha \in \mathbb{Z}_+.$$

Therefore, an application of the inverse Fourier transform shows that $\{\varphi(\cdot \oplus h_{[\alpha]}) \mid \alpha \in \mathbb{Z}_+\}$ is a Parseval frame for $V_0(\gamma)$. Also, in view of (2.1),

$$V_j(\gamma) = D^j(V_0(\gamma)), \quad \text{where} \quad Df(x) = 2^{1/2}f(Ax).$$

Observing that $\varphi_{j,\alpha} = D^j\varphi_{0,\alpha}$, we conclude that, for each $j \in \mathbb{Z}$, the system $\{\varphi_{j\,\alpha} \mid \alpha \in \mathbb{Z}_+\}$ is a Parseval frame for $V_j(\gamma)$. From this, by Proposition 1, for any $f \in V_j(\gamma)$ we have

$$f(x) = \sum_{\alpha \in \mathbb{Z}_+} \langle f, \varphi_{j,\alpha}\rangle \varphi_{j,\alpha}(x), \quad x \in G,$$

where

$$\langle f, \varphi_{j,\alpha}\rangle = 2^{j/2} \int_G f(x)\varphi(A^j x \oplus h_{[\alpha]})\,dx$$

$$= 2^{-j/2} \int\limits_E \widehat{f}(\omega)\mathbf{1}_E(A^{-1}\omega)W_\alpha(A^{-1}\omega)\,d\omega$$

$$= 2^{-j/2} \int\limits_G \widehat{f}(\omega)\chi(A^{-1}h_{[\alpha]},\omega)\,d\omega = 2^{-j/2}f(A^{-1}h_{[\alpha]}),$$

which yields (2.2). Now, let us denote by $W_j(\gamma)$ the orthogonal complement of $V_j(\gamma)$ in $V_{j+1}(\gamma)$:

$$W_j(\gamma) = V_j(\gamma)^\perp \cap V_{j+1}(\gamma), \quad j \in \mathbb{Z},$$

and let $E_j := A^{j+1}(E) \setminus A^j(E)$. Then the orthogonal direct sum of all $W_j(\gamma)$ coincides with $L^2(G)$ and, for each j, the orthogonal projection $Q_j : L^2(G) \to W_j(\gamma)$ can be defined as follows:

$$g_j = Q_j f \iff \widehat{g}_j = \widehat{f} \cdot \mathbf{1}_{E_j}, \quad f \in L^2(G).$$

Hence, for $\psi = \mathcal{D}_{A\gamma} - \mathcal{D}_\gamma$, we see that

$$\widehat{\psi}_{j\,\alpha}(\omega) = 2^{-j/2}W_\alpha(A^{-j}\omega)\mathbf{1}_{E_j}(\omega), \quad j \in \mathbb{Z}, \quad \alpha \in \mathbb{Z}_+,$$

and that, for each j, the system $\{\psi_{j\,\alpha} \mid \alpha \in \mathbb{Z}_+\}$ is a Parseval frame for $W_j(\gamma)$. The proposition is proved. □

As a consequence, we obtain the following equality for $\mathcal{D}_\gamma$:

$$\mathcal{D}_\gamma(x) = \sum_{h\in H} \mathcal{D}_\gamma(A^{-1}h)\mathcal{D}_\gamma(Ax \oplus h), \quad x \in G.$$

Indeed, since $\mathcal{D}_\gamma \in V_0(\gamma) \subset V_1(\gamma)$, we can apply (2.2) for $f = \mathcal{D}_\gamma$ and $j = 1$.

3. Examples of dyadic wavelet frames

Denote by $L^2_c(G)$ the set of all compactly supported functions in $L^2(G)$. We say that $\varphi \in L^2_c(G)$ is a *refinable function*, if it satisfies an equation of the type

$$\varphi(x) = 2\sum_{\alpha=0}^{2^n-1} a_\alpha \varphi(Ax \oplus h_{[\alpha]}). \tag{3.1}$$

The functional equation (3.1) is called the *refinement equation*. The Walsh polynomial

$$m_0(\omega) = \sum_{\alpha=0}^{2^n-1} a_\alpha W_\alpha(\omega)$$

is called the *mask* of equation (3.1) (or its solution φ). In view of (3.1), we have $\widehat{\varphi}(\omega) = m_0(A^{-1}\omega)\widehat{\varphi}(A^{-1}\omega)$ Note that if $a_0 = a_1 = 1/2$ and $a_\alpha = 0$ for all $\alpha \geq 2$, then equation (3.1) have the solution $\varphi = \mathbf{1}_{U_{n-1}}$. In particular, when $n = 1$, the Haar function φ_H satisfies this equation.

The sets

$$U_{n,\,s} := A^{-n}(h_{[s]}) \oplus A^{-n}(U), \quad 0 \leq s \leq 2^n - 1,$$

are cosets of the subgroup $A^{-n}(U)$ in the group U. For every $0 \leq \alpha \leq 2^n - 1$ the Walsh function $W_\alpha(\cdot)$ is constant on each set $U_{n,\,s}$. The coefficients of equation (3.1) are related to the values $b_s = m_0(A^{-n}(h_{[s]}))$ by means of the discrete Walsh transform

$$a_\alpha = \frac{1}{2^n} \sum_{s=0}^{2^n-1} b_s W_\alpha(A^{-n} h_{[s]}), \quad 0 \leq \alpha \leq 2^n - 1,$$

which can be realized by the fast algorithm (e.g., [4,6]). Thus, any choice of the values b_s defines also the coefficients a_α in (3.1).

We recall that a collection of closed subspaces $V_j \subset L^2(G)$, $j \in \mathbb{Z}$, is called a *multiresolution analysis* (*an MRA*) in $L^2(G)$ if the following hold:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(ii) $\overline{\bigcup V_j} = L^2(G)$ and $\bigcap V_j = \{0\}$;

(iii) $f(\cdot) \in V_j \iff f(A\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(iv) there is a function $\varphi \in L^2(G)$ such that the system $\{\varphi(\cdot \oplus h) \mid h \in H\}$ is an orthonormal basis of V_0.

The function φ in condition (iv) is called a *scaling function* in $L^2(G)$.

We say that *a function φ generates an MRA in* $L^2(G)$ if the family $\{\varphi(\cdot \oplus h) \mid h \in H\}$ is an orthonormal system in $L^2(G)$ and, in addition, the family of subspaces

$$V_j = \mathrm{clos}_{L^2(G)} \mathrm{span}\, \{\varphi_{j,\alpha} \mid \alpha \in \mathbb{Z}_+\}, \quad j \in \mathbb{Z},$$

is the MRA in $L^2(G)$. If a function φ generates an MRA in $L^2(G)$, then it is a scaling function in $L^2(G)$. In this case the system $\{\varphi_{j,\alpha} \mid \alpha \in \mathbb{Z}_+\}$ is an orthonormal basis of V_j for every $j \in \mathbb{Z}$ and one can define an *orthogonal wavelet* ψ in such a way that $\{2^{j/2}\psi(A^j \cdot \oplus h) \mid j \in \mathbb{Z},\, h \in H\}$ is an orthonormal basis of $L^2(G)$.

Example 3.1. Let $n = 2$ and

$$b_0 = 1,\ b_1 = a,\ b_2 = 0,\ b_3 = b,$$

where $|\,a|^2 + |\,b|^2 = 1$. Then in (3.1) we have

$$a_0 = (1 + a + b)/4, \quad a_1 = (1 + a - b)/4,$$

$$a_2 = (1 - a - b)/4, \quad a_3 = (1 - a + b)/4.$$

Denote by φ the corresponding solution of equation (3.1). If $a \neq 0$, then φ generates an MRA in $L^2(G)$ (see [14]). In particular, for $a = 1$ and $a = -1$ the Haar function $\mathbf{1}_U$ and the displaced Haar function $\mathbf{1}_U(\cdot \oplus h_{[1]})$ are obtained respectively. If $b = 1$ and $b = -1$, we return to Example 1.2. If $0 < |\,a| < 1$, then φ can be written in the form

$$\varphi(x) = (1/2)\mathbf{1}_U(A^{-1}x)\Big(1 + a\sum_{j=0}^{\infty} b^j W_{2^{j+1}-1}(A^{-1}x)\Big), \quad x \in G.$$

In this case,

$$\psi(x) = 2a_0\varphi(Ax \oplus h_{[1]}) - 2a_1\varphi(Ax) + 2a_2\varphi(Ax \oplus h_{[3]}) - 2a_3\varphi(Ax \oplus h_{[2]}).$$

Also, when $0 < |\,b| < 1/2$, the system $\{\psi_{j,\alpha}\}$ is an unconditional basis in all spaces $L^p(G)$, $1 < p < \infty$. Moreover, the relevant wavelets on the line may be identified as multiwavelets consisting of piecewise fractal functions, in the sense of Massopust (see [15, 16]).

Suppose that M either is the union of some of the sets $U_{n-1,s}$, $1 \le s \le 2^{n-1} - 1$, or coincides with one of these sets. Then we define

$$T(M) := \{A^{-1}(\omega) \mid \omega \in M\} \cup \{e_0 \oplus A^{-1}(\omega) \mid \omega \in M\}.$$

A set M is said to be *blocking* (for a mask m_0) if it satisfies the condition

$$T(M) \subset M \cup \{\omega \in U \mid m_0(\omega) = 0\}.$$

A compact subset E of G is said to be *congruent to U modulo H* if $\mu(E) = 1$ and, for each $\omega \in E$, there is an element $h \in H$ such that $\omega \oplus h \in U$. We say that a mask m_0 satisfies the *modified Cohen condition*, if there exists a compact subset E of G containing a neighbourhood of θ such that E congruent to U modulo H and $\inf_{j\in\mathbb{N}} \inf_{\omega\in E} |m_0(A^{-j}\omega)| > 0$.

Theorem 3.1. *Let $\varphi \in L^2_c(G)$ be a solution of refinement equation* (3.1) *such that $\widehat{\varphi}(\theta) = 1$. Suppose that its mask m_0 satisfies the condition*

$$|\,m_0(\omega)|^2 + |\,m_0(\omega \oplus e_0)|^2 = 1 \quad \textit{for all} \quad \omega \in G.$$

Then the following are equivalent:

(*a*) *φ generates an MRA in $L^2(G)$;*

(*b*) *m_0 satisfies the modified Cohen's condition;*

(*c*) *m_0 has no blocking sets.*

The proof of Theorem 3.1 is given in [17] (cf. [18, Theorem 3]) where an algorithm for the expansions of a dyadic scaling function φ in Walsh series is also discussed.

Now, we set

$$D_\psi(\omega) := \sum_{j\in\mathbb{Z}} |\,\widehat{\psi}(A^{-j}\omega)|^2,$$

$$M_{s,\psi} := \sup_{\omega\in G} \sum_{j\in\mathbb{Z}} |\,\widehat{\psi}(A^{-j}\omega)||\,\widehat{\psi}(A^{-j}\omega \oplus h_{[s]})|, \quad s\in\mathbb{N}.$$

Observe that $D_\psi(\omega) = D_\psi(A\omega)$ for all $\omega \in G$, and that supremum in the definition of $M_{s,\psi}$ is invariant under the transform $\omega \mapsto A\omega$, so that this supremum can be taken over $\omega \in U_{1,0}$.

Theorem 3.2. *Let $\psi \in L^2(G)$ be such that*

$$c_0^{(1)} := \operatorname*{ess\,inf}_{\omega\in G} D_\psi(\omega) - \sum_{s\in\mathbb{N}} M_{s,\psi} > 0,$$

and

$$c_1^{(1)} := \operatorname*{ess\,sup}_{\omega\in G} D_\psi(\omega) + \sum_{s\in\mathbb{N}} M_{s,\psi} < \infty.$$

Then $\{\psi_{j,\alpha}\}$ is a frame with frame bounds $c_0^{(1)}$ and $c_1^{(1)}$.

Notice that Theorem 3.2 includes the Daubechies type "admissible condition" (cf. [19, Section 3.3.2], [20]).

Let

$$g_{l,s}(x) := 2^{-s}\mathbf{1}_U(A^{-s}x)w_l(A^{-s}x), \quad l,s\in\mathbb{Z}_+.$$

It is easy to check that $\widehat{g}_{l,s} = \mathbf{1}_{U_{l,s}}$, where, as before, $U_{l,s} = A^{-l}((h_{[s]} \oplus U)$. In particular, $g_{1,0}$ coincides with the Haar wavelet ψ_H. According to [21, 22], any compactly supported orthogonal wavelet in $L^2(G)$ can be expanded in the gap series with respect to $g_{l,s}$ or coincides with a finite linear combination of $g_{l,s}$ (several examples of such wavelets are given in [23]).

Using Theorem 3.2 and the functions $g_{l,s}$, we can construct frames for $L^2(G)$. Note that similar examples of frames for the space $L^2(\mathbb{R}_+)$ are given in [24].

Example 3.2. Suppose that $\psi = g_{l,s}$ with $l \in \mathbb{N}$, $s \in \mathbb{Z}_+$. Then for any $\alpha \in \mathbb{N}$ the supports of $\widehat{\psi}(A^{-j}\omega)$ and $\widehat{\psi}(A^{-j}\omega \oplus h_{[\alpha]})$ are disjoint. Since

$$\operatorname*{ess\,inf}_{\omega \in U_{1,0}} D_\psi(\omega) = \operatorname*{ess\,sup}_{\omega \in U_{1,0}} D_\psi(\omega) = 1,$$

we see that $c_1^{(0)} = c_1^{(1)} = 1$. Therefore, by Theorem 3.2, $\{\psi_{j,\alpha}\}$ is a Parseval frame for $L^2(G)$.

Example 3.3. Let us assume that

$$\psi(x) = g_{1,4}(x) + \nu g_{1,1}(x),$$

where ν is a positive parameter. Then, for every $\omega \in G$,

$$\widehat{\psi}(\omega) = \mathbf{1}_{U_{1,4}}(\omega) + \nu \mathbf{1}_{U_{1,1}}(\omega), \quad |\widehat{\psi}(\omega)|^2 = \mathbf{1}_{U_{1,4}}(\omega) + \nu^2 \mathbf{1}_{U_{1,1}}(\omega),$$

and

$$\widehat{\psi}(A^{-j}\omega)\widehat{\psi}(A^{-j}\omega \oplus h_{[\alpha]}) = 0 \quad \text{for all } j \in \mathbb{Z},\ \alpha \in \mathbb{N}.$$

From these equalities we deduce that

$$c_1^{(0)} = \nu^2, \quad c_1^{(1)} = 1 + \nu^2, \quad c_1^{(1)}/c_1^{(0)} = 1 + \frac{1}{\nu^2}.$$

Thus, $\{\psi_{j,\alpha}\}$ tends to the tight frame when $\nu \to \infty$.

Example 3.4. Given a non zero element $b \in G$ we let $\beta = \lambda(b)$ and denote by $g_{l,s}^{(\beta)}$ a function in $L^2(G)$ such that $\widehat{g}_{l,s}^{(\beta)} = \mathbf{1}_{U_{l,s}^{(\beta)}}$, where $U_{l,s}^{(\beta)} := b^{-s}(h_{[l]} + U)$, $l \in \mathbb{N}$, $s \in \mathbb{Z}_+$. In particular, for $b = e_{-1}$ we have $g_{l,s}^{(\beta)} = g_{l,s}^{(1)} = g_{l,s}$. As in Example 5, we obtain that if $\psi = g_{l,s}^{(\beta)}$, $\beta > 1/2$, then $\{\psi_{j,\alpha}\}$ is a Parseval frame for $L^2(G)$. Moreover, using Theorem 3.2, we derive the following frames for $L^2(G)$ with frame bounds $c_1^{(0)}$ and $c_1^{(1)}$:

1) $\psi(x) = g_{1,4}^{(3/2)}(x) + 2g_{2,1}^{(3/2)}(x)$, $\quad c_1^{(0)} = 4$, $\quad c_1^{(1)} = 5$,
$c_1^{(1)}/c_1^{(0)} = 1,25$,

2) $\psi(x) = g_{3,1}^{(1)}(x) + 4g_{1,1}^{(0,85)}(x)$, $\quad c_1^{(0)} = 9$, $\quad c_1^{(1)} = 25$,
$c_1^{(1)}/c_1^{(0)} = 2,777\ldots$,

3) $\psi(x) = g_{2,1}^{(0,5)}(x) + 3g_{1,1}^{(0,8)}(x)$, $\quad c_1^{(0)} = 4$, $\quad c_1^{(1)} = 22$,
$c_1^{(1)}/c_1^{(0)} = 5,5$.

Given functions $\psi^1, \dots, \psi^N \in L^2(G)$ and a number $s \in \{0, 1, 2\}$ we let

$$\psi^{\nu,\, s}_{j,\, \alpha}(x) := 2^{j/2}\psi^{\nu}(A^j x \oplus A^{-s}h_{[\alpha]}), \quad \nu \in \{1, \dots, N\},\ j \in \mathbb{Z},\ \alpha \in \mathbb{Z}_+.$$

The set $\{\psi^{\nu,\, s}_{j,\, \alpha}\}$ is called *framelet* for $L^2(G)$ if there exist positive constants c_0, c_1 such that, for every $f \in L^2(G)$,

$$c_0\|f\|^2 \le \sum_{\nu=1}^{N}\sum_{j,\, \alpha} |\langle f\,, \psi^{\nu,\, s}_{j,\, \alpha}\rangle|^2 \le c_1\|f\|^2.$$

Let us denote

$$M^{\nu}(h, s) := \sup_{\omega \in G}\sum_{j} |\,\widehat{\psi}^{\nu}(A^{-j}\omega)||\widehat{\psi}^{\nu,\, s}(A^{-j}\omega \oplus A^s h)|,$$

$$R^{(N)}(s) := \sum_{\nu=1}^{N}\sum_{h \in H} M^{\nu}(h, s),$$

$$c_0^{(N)}(s) := \operatorname*{ess\,inf}_{\omega \in G}\sum_{\nu=1}^{N}\sum_{j} |\,\widehat{\psi}^{\nu,\, s}(A^{-j}\omega)|^2 - R^{(N)}(s),$$

$$c_1^{(N)}(s) := \operatorname*{ess\,sup}_{\omega \in G}\sum_{\nu=1}^{N}\sum_{j} |\,\widehat{\psi}^{\nu}(A^{-j}\omega)|^2 + R^{(N)}(s).$$

Theorem 3.3 (cf. [19, Section 3.3.4]). *Let $\psi^1, \dots, \psi^N \in L^2(G)$ be such that $c_0^{(N)}(s) > 0$ and $c_1^{(N)}(s) < \infty$. Then $\{\psi^{\nu,\, s}_{j,\, \alpha}\}$ is framelet for $L^2(G)$ and*

$$2^s c_0^{(N)}(s)\|f\|^2 \le \sum_{\nu=1}^{N}\sum_{j,\, \alpha} |\langle f\,, \psi^{\nu,\, s}_{j,\, \alpha}\rangle|^2 \le 2^s c_1^{(N)}(s)\|f\|^2$$

for all $f \in L^2(G)$.

Theorem 3.4 (cf. [13, Theorem 1.8.11]). *Let $\varphi \in L^2_c(G)$ be a solution of refinement equation* (3.1) *and let its mask m_0 satisfies the condition*

$$|\,m_0(\omega)|^2 + |m_0(\omega \oplus e_0)|^2 = 1 \quad \textit{for all} \quad \omega \in G. \tag{3.2}$$

Suppose that φ is continuous at θ, $\widehat{\varphi}(\theta) \neq 0$, and that ψ is defined by $\widehat{\psi}(\omega) = m_1(A^{-1}\omega)\widehat{\varphi}(A^{-1}\omega)$, where m_1 is a Walsh polynomial such that the matrix

$$\begin{pmatrix} m_0(\omega) & m_1(\omega) \\ m_0(\omega \oplus e_0) & m_1(\omega \oplus e_0) \end{pmatrix}$$

is unitary for all $\omega \in G$. Then $\{\psi_{j,\, \alpha}\}$ is a tight frame with frame bound $|\,\widehat{\varphi}(\theta)|^2$.

In particular, we can choose

$$\psi(x) = 2 \sum_{\alpha=0}^{2^n-1} (-1)^{\alpha} a_{2^n-1-\alpha} \varphi(Ax \oplus h_{[\alpha]}), \quad x \in G. \qquad (3.3)$$

Observe that when (3.2) holds and m_0 has no blocking sets, Theorem 1 implies that ψ is an orthogonal wavelet for $L^2(G)$. Thus, if m_0 has a blocking set, than $\{\psi_{j,\alpha}\}$ will be a tight frame which is not a basis in $L^2(G)$.

Example 3.5. Let us choose numbers a, b, c and α, β, γ such that

$$|a|^2 + |\alpha|^2 = |b|^2 + |\beta|^2 = |c|^2 + |\gamma|^2 = 1.$$

Suppose that φ satisfies the refinement equation

$$\varphi(x) = 2 \sum_{\alpha=0}^{7} a_\alpha \varphi(Ax \oplus h_{[\alpha]}).$$

with the coefficients

$$a_0 = \frac{1}{8}(1 + a + b + c + \alpha + \beta + \gamma),$$

$$a_1 = \frac{1}{8}(1 + a + b + c - \alpha - \beta - \gamma),$$

$$a_2 = \frac{1}{8}(1 + a - b - c + \alpha - \beta - \gamma),$$

$$a_3 = \frac{1}{8}(1 + a - b - c - \alpha + \beta + \gamma),$$

$$a_4 = \frac{1}{8}(1 - a + b - c - \alpha + \beta - \gamma),$$

$$a_5 = \frac{1}{8}(1 - a + b - c + \alpha - \beta + \gamma),$$

$$a_6 = \frac{1}{8}(1 - a - b + c - \alpha - \beta + \gamma),$$

$$a_7 = \frac{1}{8}(1 - a - b + c + \alpha + \beta - \gamma).$$

The blocking sets for the mask

$$m(\omega) = \sum_{\alpha=0}^{7} a_\alpha W_\alpha(\omega)$$

are: 1) $U_{1,1} \cup U_{2,1}$ for $a = 0$, 2) $U_{2,1} \cup U_{2,2}$ for $a = \beta = 0$, 3) $U_{2,3}$ for $c = 0$, 4) $U_{1,1}$ for $b = c = 0$. If $abc \neq 0$ then the modified Cohen condition is fulfilled for $E = U$, and when $a \neq 0$, $b = 0$, $c \neq 0$ it holds for $E = A(U_{3,0} \cup U_{3,1} \cup U_{3,3} \cup U_{3,6})$. Hence, by Theorem 3.1, if a and c distinct from zero, then φ generate an MRA in $L^2(G)$. Now, let ψ be given by (3.3). Then, applying Theorem 3.4, we conclude that if $ac \neq 0$ then $\{\psi_{j,\alpha}\}$ is an orthonormal basis in $L^2(G)$, and if $ac = 0$ then $\{\psi_{j,\alpha}\}$ is a Parseval frame for $L^2(G)$.

Remark 3.1. The dyadic modulus of continuity of the scaling function φ satisfying equation (3.1) is defined by the equality

$$\omega(\varphi, \delta) := \sup\{|\,\varphi(x \oplus y) - \varphi(x)| \,:\, x, y \in G,\, \lambda(y) \in [0, \delta)\}, \quad \delta > 0.$$

If φ satisfies $\omega(\varphi, 2^{-j}) \leq C 2^{-\alpha j}$, $j \in \mathbb{N}$, for some $\alpha > 0$, then there exists a constant $C(\varphi, \alpha)$, such that

$$\omega(\varphi, \delta) \leq C(\varphi, \alpha)\, \delta^{\alpha}. \tag{3.4}$$

Denote by α_φ the supremum for the set of all values $\alpha > 0$ for which inequality (3.4) holds. According to [21], if $n = 2$ then $\alpha_\varphi = \log_2(1/|b\,|)$, where b as in Example 1.2. For the cases $n = 3$ and $n = 4$ some values of α_φ are calculated in [23].

Remark 3.2. In [25], sevaral discrete p-dyadic wavelet bases for $\ell^2(\mathbb{Z}_N)$ are defined by finite collections of parameters. Similar bases for locally compact Vilenkin groups were studied in [17] and [26]. Note that the values of parameters in [24] are universal in the sense that they occur not only in the construction of orthogonal wavelets in $\ell^2(\mathbb{Z}_N)$, but also in the space $\ell^2(\mathbb{Z}_+)$. At the same time, the corresponding wavelet constructions on the Cantor and Vilenkin groups (as well as on the half-line $\mathbb{R}_+$; see [27]) requires an additional constraint related to the requirement that the masks have no blocking sets. The great freedom in the construction of wavelets in $\ell^2(\mathbb{Z}_N)$ extends the range of applications of the adaptive signal-approximation methods; some numerical experiments comparing discrete dyadic wavelets with the Haar and Daubechies wavelets in an image processing scheme are discussed in [28] and [29].

References

[1] G. H. Agaev, N. J. Vilenkin, G. M. Dzhafarli, and A. Rubinstein, *Multiplicative systems of functions and harmonic analysis on 0-dimensional groups*, Baku: ELM, 1981 (in Russian).

[2] L. S. Pontryagin, *Topological groups*, London: Oxford Univ. Press, 1939.

[3] R. E. Edwards, *Fourier series. A modern introduction.* Vol. 2, Berlin: Springer Verlag, 1982.

[4] F. Schipp, W. R. Wade, P. Simon, *Walsh series: An introduction to dyadic harmonic analysis*, N.Y.: Adam Hilger, 1990.

[5] B. I. Golubov, *Elements of dyadic analysis.* M.: URSS, 2007. (Russian)

[6] B. I. Golubov, A. V. Efimov, and V. A. Skvortsov, *Walsh series and transforms: theory and applications*, M.: URSS, 2008 (in Russian).

[7] W. Y. Su, *Gibbs derivatives — the development over 40 years in China*, in "Walsh and Dyadic Analysis", Proc. of the Workshop Dedicated to the Memory of Edmund Gibbs, R.S. Stanković, Ed., Niš, Serbia, Oct. 18–19, 2007, pp. 15–30.

[8] R. S. Stanković, J. Astola, *Remarks on the development and recent results in the theory of Gibbs derivatives* // Facta Univ., Ser. Electron. Energ., **21** (2008), No. 3, 349–364.

[9] G. Gát, *Convergence and divergence of Fejér means of Fourier series on one and two-dimensional Walsh and Vilenkin groups* // Facta Univ., Ser. Electron. Energ., **21** (2008), No. 3, 391–307.

[10] W. C. Lang, *Orthogonal wavelets on the Cantor dyadic group* // SIAM J. Math. Anal., **27** (1996), 305–312.

[11] J. J. Benedetto, E. J. King, *Smooth functions associated with wavelet sets on $\mathbb{R}^d$, $d \geq 1$, and frame bounds gaps* // Acta Appl. Math., **107** (2009), 121–142.

[12] J. J. Benedetto, R. L. Benedetto, *A wavelet theory for local fields and related groups* // J. Geometric Analysis, **14** (2004), 423–456.

[13] I. Ya. Novikov, V. Yu. Protasov, and M. A. Skopina, *Wavelet Theory*, Translations of Mathematical Monographs 239, Providence, RI: American Mathematical Society (AMS), 2011.

[14] W. C. Lang, *Orthogonal wavelets on the Cantor dyadic group* // SIAM J. Math. Anal., **27** (1996), 305–312.

[15] W. C. Lang, *Fractal multiwavelets related to the Cantor dyadic group* // Intern. J. Math. and Math. Sci., **21** (1998), 307–317.

[16] W. C. Lang, *Wavelet analysis on the Cantor dyadic group* // Houston J. Math., **24** (1998), 533–544.

[17] Yu. A. Farkov, *Orthogonal wavelets on direct products of cyclic groups* // Mat. Zametki, **82** (2007), No. 6, 934–952; English transl.: Math. Notes, **82** (2007), No. 6.

[18] V. Yu. Protasov, Yu. A. Farkov, *Dyadic wavelets and refinable functions on a half-line* // Mat. Sbornik, **197**(2006), No. 10, 129–160; English transl.: Sbornik: Mathematics, **197** (2006), No. 10, 1529–1558.

[19] I. Daubechies, *Ten Lectures on Wavelets*, Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, 1992.

[20] C. K. Chui, W. He, *Compactly supported tight frames associated with refinable functions* // Appl. and Comp. Harm. Anal., **8** (2000), 293–319.

[21] Yu. A. Farkov, *Orthogonal wavelets with compact support on locally compact abelian groups* // Izvestiya RAN: Ser. Mat., **69** (2005), No. 3, 193–220; English transl.: Izvestiya: Mathematics, **69** (2005), No. 3, 623–650.

[22] Yu. A. Farkov, *Multiresolution analysis and wavelets on Vilenkin groups* // Facta Univers. (NIŠ). Ser.: Elec. Energ., **21** (2008), No. 3, 11–27.

[23] E. A. Rodionov, Yu. A. Farkov, *Estimates of the smoothness of dyadic orthogonal wavelets of Daubechies type* // Math. Notes, **86** (2009), No. 3, 392–406.

[24] Yu. A. Farkov, *Wavelets and frames based on Walsh-Dirichlet type kernels* // Comm. Math. Appl., **1** (2010), 27–46.

[25] Yu. A. Farkov, *Discrete wavelets and the Vilenkin–Chrestenson transform* // Mat. Zametki **89** (2011), No. 6, 914—928; [Math. Notes **89** (2011), No. 6].

[26] Yu. A. Farkov, E. A. Rodionov, *Algorithms for wavelet construction on Vilenkin groups* // p-Adic Numbers, Ultrametric Analysis and Applications, **3** (2011), No. 3, 181-–195.

[27] Yu. A. Farkov, *On wavelets related to the Walsh series* // J. Approx. Theory, **161** (2009), No. 1, 259–279.

[28] Yu. A. Farkov, A. Yu. Maksimov and S. A. Stroganov, *On biorthogonal wavelets related to the Walsh functions* // Int. J. Wavelets Multiresolut. Inf. Process, **9** (2011), No. 3, 44–58.

[29] Yu. A. Farkov, S. A. Stroganov, *On discrete dyadic wavelets for image processing* // Izv. VUZ. Matematika, **7** (2011), 57–66; [Russian Mathematics (Iz. VUZ), **7**, (2011)].

Contact information

Yuri A. Farkov
Russian State Geological
Prospecting University
Miklouho-Maklay's str., 23
117997, Moscow
Russia
E-Mail: farkov@list.ru