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RESEARCH ARTICLES

Algorithms for Wavelet Construction on Vilenkin Groups*

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Abstract—In this paper, some algorithms for constructing orthogonal and biorthogonal compactly supported wavelets on Vilenkin groups are suggested. As application, several examples of p-adic wavelets, which correspond to the refinable functions presented recently by the first author, are given.

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1. INTRODUCTION

The p-adic Vilenkin group G_p can be defined as the weak direct product of a countable set of cyclic groups of order p (in the case of $p = 2$, G_p is isomorphic to the Cantor dyadic group). It is well known that the characters of G_p are the generalized Walsh functions (e.g., [1, 2]). Some constructions of orthogonal wavelets on Vilenkin groups were studied in [3–7]. In this article, we offer new algorithms for constructing orthogonal and biorthogonal wavelet bases in $L^2(G_p)$. To illustrate these, we build examples of wavelets with compact support on G_p which correspond to the refinable functions presented recently by the first author in [8, 9]. In orthogonal wavelet construction on G_p the mask of refinable function is a generalized Walsh polynomial (see Example 1). In this case, we use the discrete Vilenkin-Chrestenson transform and find an appropriate sequence of unitary transforms to solve the problem of completing an unitary matrix with the first row given. More details on this problem for wavelet construction on $\mathbb{R}^d$ can be found in [10–12].

We now introduce some notations and state preliminary results (cf. [6, 9]). Let us recall that $G = G_p$ consists of sequences of the form

$$x = (x_j) = (\dots, 0, 0, x_k, x_{k+1}, x_{k+2}, \dots),$$

where $x_j \in \{0, 1, \dots, p-1\}$ for $j \in \mathbb{Z}$ and $x_j = 0$ for $j < k = k(x)$. The group operation on G is denoted by $\oplus$ and defined as coordinatewise addition modulo p:

$$(z_j) = (x_j) \oplus (y_j) \iff z_j = x_j + y_j \,(\mathrm{mod}\, p) \quad \text{for all} \quad j \in \mathbb{Z};$$

the topology on G is determined by the basis of neighborhoods of zero:

$$U_l = \{(x_j) \in G \mid x_j = 0 \text{ for } j \leqslant l\}, \quad l \in \mathbb{Z}.$$

We denote the inverse operation of $\oplus$ by $\ominus$ (so that $x \ominus x = \theta$, where θ is the zero sequence). As in [9], define U as a subgroup of G with the set of elements U_0.

The Lebesgue spaces $L^q(G)$ are considered with respect to the Haar measure μ defined on the Borel subsets of G and normalized by the condition $\mu(U) = 1$. By $(\cdot, \cdot)$ we denote the inner product on $L^2(G)$. The dual group of G is denoted by G^* and consists of sequences of the form

$$\omega = (\omega_j) = (\dots, 0, 0, \omega_k, \omega_{k+1}, \omega_{k+2}, \dots),$$

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where $\omega_j \in \{0, 1, \dots, p-1\}$ for $j \in \mathbb{Z}$ and $\omega_j = 0$ for $j < k = k(\omega)$. Addition and subtraction, neighborhoods of zero $\{U_l^*\}$ and the Haar measure μ^* are defined for G^* in the same way as for G. Each character of the group G can be represented as

$$\chi(x, \omega) = \exp\left(\frac{2\pi i}{p} \sum_{j \in \mathbb{Z}} x_j \omega_{1-j}\right), \qquad x \in G,$$

for some $\omega \in G^*$.

Consider the discrete subgroup $H = \{(x_j) \in G \mid x_j = 0 \text{ for } j > 0\}$ in G and define an automorphism $A \in \operatorname{Aut} G$ as $(Ax)_j = x_{j+1}$. Notice that the quotient group $H/A(H)$ contains p elements, and the annihilator $H^\perp$ of H consists of those sequences (ω_j) in which $\omega_j = 0$ for $j > 0$.

The mapping $\lambda : G \to [0, +\infty)$ is defined by

$$\lambda(x) = \sum_{j \in \mathbb{Z}} x_j p^{-j}, \qquad x = (x_j) \in G.$$

It is clear that the image of the subgroup H under λ is the set of nonnegative integers: $\lambda(H) = \mathbb{Z}_+$. For each $\alpha \in \mathbb{Z}_+$, let $h_{[\alpha]}$ denote the element of H such that $\lambda(h_{[\alpha]}) = \alpha$; in particular, $h_{[0]} = \theta$. A mapping $\lambda^* : G^* \to [0, +\infty)$, an automorphism $B \in \operatorname{Aut} G^*$, a subgroup U^* in G^*, and elements $\omega_{[\alpha]}$ are defined by analogy with λ, A, U, and $h_{[\alpha]}$, respectively.

Generalized Walsh functions for the group G can be defined as

$$W_\alpha(x) = \chi(x, \omega_{[\alpha]}), \qquad \alpha \in \mathbb{Z}_+,\ x \in G.$$

These functions are continuous on G and satisfy the orthogonality relations

$$\int_U W_\alpha(x) \overline{W_\beta(x)} d\mu(x) = \delta_{\alpha,\beta}, \qquad \alpha, \beta \in \mathbb{Z}_+,$$

where $\delta_{\alpha,\beta}$ denotes the Kronecker delta. It is known that the system $\{W_\alpha\}$ is complete in $L^2(U)$. The corresponding system for the group G^* is defined by

$$W_\alpha^*(\omega) = \chi(h_{[\alpha]}, \omega), \qquad \alpha \in \mathbb{Z}_+,\ \omega \in G^*.$$

The system $\{W_\alpha^*\}$ is an orthonormal basis in $L^2(U^*)$.

For each function $f \in L^1(G) \cap L^2(G)$ its Fourier transform $\widehat{f}$,

$$\widehat{f}(\omega) = \int_G f(x) \overline{\chi(x, \omega)} d\mu(x), \quad \omega \in G,$$

belongs to the space $L^2(G)$. The Fourier operator

$$\mathcal{F} : L^1(G) \cap L^2(G) \to L^2(G), \quad \mathcal{F} f = \widehat{f},$$

admits a standard extension to the whole space $L^2(G)$.

We will denote by $L_c^2(G)$ the set of functions in $L^2(G)$ with compact support. A function $\varphi \in L_c^2(G)$ is called a *refinable function* if it satisfies an equation of the form

$$\varphi(x) = p \sum_{\alpha=0}^{p^n-1} a_\alpha \varphi(Ax \ominus h_{[\alpha]}), \quad x \in G, \tag{1.1}$$

where a_α are complex coefficients.

We denote the characteristic function of a set $E \subset G$ by $\mathbf{1}_E$. If $a_0 = \dots = a_{p-1} = 1/p$ and $a_\alpha = 0$ for all $\alpha \geqslant p$, then the function $\varphi = \mathbf{1}_{U_{n-1}}$ is a solution of equation (1.1). The corresponding orthogonal wavelets in $L^2(G)$ have the form

$$\psi^{(\nu)}(x) = \sum_{\alpha=0}^{p-1} \varepsilon_p^{\nu\alpha} \varphi(Ax \ominus h_{[\alpha]}), \quad \nu = 1, \dots, p-1,$$

where $\varepsilon_p = \exp(2\pi i/p)$ (cf. [7], Section 3; 13, Theorem 2).

The functional equation (1.1) is known as the *refinement equation*. Applying the Fourier transform, we can write this equation as

$$\widehat{\varphi}(\omega) = m(B^{-1}\omega)\widehat{\varphi}(B^{-1}\omega), \tag{1.2}$$

where

$$m(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W_\alpha^*(\omega)} \tag{1.3}$$

is a generalized Walsh polynomial, which is called the *mask* of the refinable function φ.

The sets

$$U_{n,s}^* = B^{-n}(\omega_{[s]}) \oplus B^{-n}(U^*), \quad 0 \leqslant s \leqslant p^n - 1, \tag{1.4}$$

are cosets of the subgroup $B^{-n}(U^*)$ in the group U^*. Each function W_α^* with $0 \leqslant \alpha \leqslant p^n - 1$ is constant on sets (1.4). The coefficients of the refinement equation (1.1) are related to the values b_s of the mask m on $U_{n,s}^*$ by the direct and inverse discrete Vilenkin-Chrestenson transforms:

$$a_\alpha = \frac{1}{p^n} \sum_{s=0}^{p^n-1} b_s W_\alpha^*(B^{-n}\omega_{[s]}), \quad 0 \leqslant \alpha \leqslant p^n - 1, \tag{1.5}$$

$$b_s = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W_\alpha^*(B^{-n}\omega_{[s]})}, \qquad 0 \leqslant s \leqslant p^n - 1; \tag{1.6}$$

see [1, 2, 14] for the corresponding fast algorithms.

As consequence of [7], Theorem 1, we have the following

PROPOSITION 1.1. *If a function* $\varphi \in L_c^2(G)$ *satisfies equation* (1.1) *and* $\widehat{\varphi}(\theta) = 1$, *then*

$$\sum_{\alpha=0}^{p^n-1} a_\alpha = 1 \quad \textit{and} \quad \operatorname{supp}\varphi \subset U_{1-n}.$$

In the space $L_c^2(G)$ *this solution of equation* (1.1) *is unique, is given by*

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(B^{-j}\omega) \tag{1.7}$$

and satisfies the partition-of-unity property:

$$\sum\nolimits_{h\in H} \varphi(x \oplus h) = 1 \quad \textit{for a.e.} \quad x \in G.$$

For $l \in \{0, 1, \ldots, p-1\}$, let δ_l denote the sequence $\omega = (\omega_j)$ in which $\omega_1 = l$ and $\omega_j = 0$ for $j \neq 1$ (in particular, $\delta_0 = \theta$). Suppose that a function $\varphi \in L_c^2(G)$ is a solution of the refinement equation (1.1) and its mask satisfies the conditions

$$m(\theta) = 1 \quad \text{and} \quad \sum_{l=0}^{p-1} |\, m(\omega \oplus \delta_l)|^2 = 1, \quad \omega \in G^*. \tag{1.8}$$

Then, as shown in [5], we have

$$\varphi(x) = (1/p^{n-1})\mathbf{1}_U(A^{1-n}x)(1 + \sum_{l\in\mathbb{N}(p,n)} d_l[m]W_l(A^{1-n}x)), \quad x \in G, \tag{1.9}$$

where $\mathbb{N}(p,n)$ and $d_l[m]$ are defined as follows. Let us represent each $l \in \mathbb{Z}_+$ in the form of a p-ary expansion

$$l = \sum_{j=0}^{k} \mu_j p^j, \quad \mu_j \in \{0,1,\dots,p-1\}, \quad \mu_k \neq 0, \quad k = k(l) \in \mathbb{Z}_+,$$

and denote the set of all positive integers $l \geqslant p^{n-1}$ for which the ordered sets $(\mu_j, \mu_{j+1}, \dots, \mu_{j+n-1})$ of the coefficients of (1.9) do not contain

$$(0,0,\dots,0,1), (0,0,\dots,0,2), \dots, (0,0,\dots,0,p-1)$$

by $\mathbb{N}_0(p,n)$. Then $\mathbb{N}(p,n) = \{1,2,\dots,p^{n-1}-1\} \cup \mathbb{N}_0(p,n)$. Let

$$\gamma(i_1,i_2,\dots,i_n) = b_s, \qquad s = i_1 p^0 + i_2 p^1 + \dots + i_n p^{n-1}, \quad i_j \in \{0,1,\dots,p-1\},$$

where b_s are defined by (1.6). Then

$$d_l[m] = \gamma(\mu_0,0,0,\dots,0,0) \quad \text{if} \quad k(l) = 0;$$

$$d_l[m] = \gamma(\mu_1,0,0,\dots,0,0)\gamma(\mu_0,\mu_1,0,\dots,0,0) \quad \text{if} \quad k(l) = 1;$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$d_l[m] = \gamma(\mu_k,0,0,\dots,0,0)\gamma(\mu_{k-1},\mu_k,0,\dots,0,0)\dots\gamma(\mu_0,\mu_1,\mu_2,\dots,\mu_{n-2},\mu_{n-1})$$

if $k = k(l) \geqslant n-1$.

Let $r = p^{n-1}$. Recall that the joint spectral radius of $r \times r$ complex matrices A_0, $A_1,\dots,A_{p-1}$ is defined as

$$\widehat{\rho}(A_0,A_1,\dots,A_{p-1}) := \lim_{k\to\infty} \max\{\|A_{d_1}A_{d_2}\dots A_{d_k}\|^{1/k} \,:\, d_j \in \{0,\,1,\dots,p-1\},\ 1 \leqslant j \leqslant k\},$$

where $\|\cdot\|$ is an arbitrary norm on $\mathbb{C}^{r\times r}$. In the case of $A_0 = \dots = A_{p-1}$, $\widehat{\rho}(A_0,\dots,A_{p-1})$ coincides with the spectral radius $\rho(A_0)$. The joint spectral radius of finite-dimensional linear operators $\mathcal{T}_0$, $\mathcal{T}_1,\dots,\mathcal{T}_{p-1}$ is defined as the joint spectral radius of their matrices in an arbitrary fixed basis of the corresponding linear space.

Let $\ominus_p$ denotes subtraction of integers modulo p. Given a refinement equation (1.1), we set $c_\alpha = p\,a_\alpha$ and define $(r\times r)$ matrices T_0, $T_1,\dots,T_{p-1}$ by

$$(T_0)_{i,j} = c_{(pi-p)\ominus_p(j-1)},\ (T_1)_{i,j} = c_{(pi-p+1)\ominus_p(j-1)}, \dots, (T_{p-1})_{i,j} = c_{(pi-1)\ominus_p(j-1)}$$

for $i,j \in \{1,2,\dots,r\}$. Consider the subspace

$$V := \{u = (u_1,\dots,u_r)^t \mid u_1 + \dots + u_r = 0\}$$

and denote by $\mathcal{T}_0$, $\mathcal{T}_1,\dots,\mathcal{T}_{p-1}$ the restrictions to V of the linear operators defined on the whole space $\mathbb{C}^r$ by the matrices T_0, $T_1,\dots,T_{p-1}$, respectively.

PROPOSITION 1.2 ([9]). *If the mask m of the refinement equation* (1.1) *satisfies the conditions*

$$m(\theta) = 1, \quad m(\delta_1) = m(\delta_2) = \dots = m(\delta_{p-1}) = 0,$$

and $\widehat{\rho}[m] := \widehat{\rho}(\mathcal{T}_0,\mathcal{T}_1,\dots,\mathcal{T}_{p-1}) < 1$, *then* φ *defined by* (1.9) *satisfies equation* (1.1) *and is continuous on* G. A family of closed subspaces $V_j \subset L^2(G)$, $j \in \mathbb{Z}$, is called a *multiresolution analysis* (or, briefly, an *MRA*) in $L^2(G)$, if the following conditions hold:

(i) $V_j \subset V_{j+1}$ for $j \in \mathbb{Z}$;

(ii) $\overline{\bigcup V_j} = L^2(G)$ and $\bigcap V_j = \{0\}$;

(iii) $f(\cdot) \in V_j \iff f(A\cdot) \in V_{j+1}$ for $j \in \mathbb{Z}$;

(iv) there exists a function $\varphi \in L^2(G)$ such that the system $\{\varphi(\cdot \ominus h) \mid h \in H\}$ is a Riesz basis in V_0.

Given a function $f \in L^2(G)$, we set

$$f_{j,h}(x) = p^{j/2} f(A^j x \ominus h), \quad j \in \mathbb{Z},\ h \in H.$$

We say that a *function φ generates an MRA in $L^2(G)$*, if, first, the family $\{\varphi(\cdot \ominus h) \mid h \in H\}$ is a Riesz system in $L^2(G)$ and, second, the closed subspaces $V_j = \overline{\operatorname{span}\{\varphi_{j,h} \mid h \in H\}}$ with $j \in \mathbb{Z}$ form an MRA in $L^2(G)$.

For $M \subset U^*$, we let

$$S_p M = \bigcup_{l=0}^{p-1} \Big\{ B^{-1}\omega_{[l]} + B^{-1}(\omega) \mid \omega \in M \Big\}.$$

Suppose that M either coincides with one of the sets $U^*_{n-1,\,s}$, $1 \leqslant s \leqslant p^{n-1} - 1$, or is the union of some of these sets. Such a set M is said to be *blocking set* for a mask m, if $m(\omega) = 0$ for all $\omega \in S_p M \setminus M$. According to this definition, if M is a blocking set for m, then $M \cap U^*_{n-1,0} = \emptyset$. Moreover, each mask may have only finitely many blocking sets.

Let E be a compact set in G^*. The set E is said to be *congruent to U^* modulo $H^\perp$* if $\mu^*(E) = 1$ and, for any $\omega \in E$, there exists an $h^* \in H^\perp$ such that $\omega \oplus h^* \in U^*$. The following theorem includes an analog of the well-known Cohen's criterion (see, e.g., [12], Theorem 2.5.6).

THEOREM 1.1 ([6, 7]). *Let φ be a refinable function whose mask m satisfies condition* (1.8) *and $\widehat{\varphi}(\theta) = 1$. Then the following conditions are equivalent*:

(a) *the function φ generates an MRA in $L^2(G)$*;

(b) *the mask m has no bloking sets*;

(c) *there exists a compact set E, congruent to U^* modulo $H^\perp$, containing a neighbourhood of the zero element, such that*

$$\inf_{j \in \mathbb{N}} \inf_{\omega \in E} |m(B^{-j}\omega)| > 0. \tag{1.10}$$

Following a standard approach (e.g., [10–12]), the problem of p-wavelet decomposition can be reduced into a problem of matrix extension. More precisely, the following *procedure to construct orthogonal p-wavelets in $L^2(G)$* was discussed in [7]:

1. Choose numbers b_s, $0 \leqslant s \leqslant p^n - 1$, so that

$$b_0 = 1, \quad |b_j|^2 + |b_{j+p^{n-1}}|^2 + \cdots + |b_{j+(p-1)p^{n-1}}|^2 = 1, \quad 0 \leqslant j \leqslant p^{n-1} - 1, \tag{1.11}$$

is true.

2. Compute a_α, $0 \leqslant \alpha \leqslant p^n - 1$, by (1.5) and verify that the mask

$$m_0(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W^*_\alpha(\omega)}.$$

has no blocked sets.

3. Find

$$m_\nu(\omega) = \sum_{\alpha \in \mathbb{Z}_+} a^{(\nu)}_\alpha \overline{W^*_\alpha(\omega)}, \quad 1 \leqslant \nu \leqslant p-1,$$

such that $(m_\nu(\omega \oplus \delta_k))_{\nu,k=0}^{p-1}$ is an unitary matrix.

4. Define $\psi^{(1)}, \ldots, \psi^{(p-1)}$ by the formula

$$\psi^{(\nu)}(x) = p \sum_{\alpha \in \mathbb{Z}_+} a^{(\nu)}_\alpha \varphi(Ax \ominus h_{[\alpha]}), \quad 1 \leqslant \nu \leqslant p-1.$$

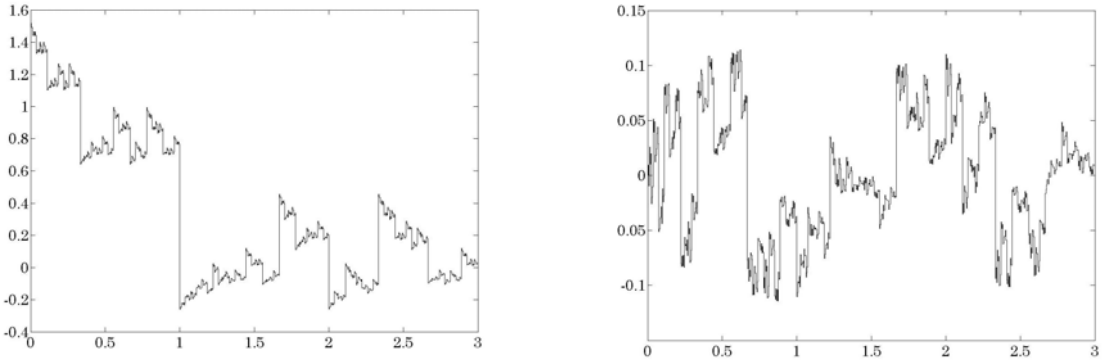

Fig. 1. Re Φ (left) and Im Φ (right) from Example 1.

Thus, we can obtain the functions

$$\psi_{j,h}^{(\nu)}(x) = p^{j/2}\psi^{(\nu)}(A^j x \ominus h), \quad 1 \leqslant \nu \leqslant p-1,\ j \in \mathbb{Z},\ h \in H,$$

which form an orthonormal basis in $L^2(G)$. In the next section, we present a new algorithm for matrix extension in the step 3 of this procedure.

2. AN ALGORITHM FOR MATRIX EXTENSION

For $0 \leqslant s, \alpha \leqslant p^n - 1$, let us choose numbers b_s as in (1.11) and compute a_α by (1.5). Then we define the mask

$$m_0(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha^{(0)} \overline{W_\alpha^*(\omega)},$$

where $a_\alpha^{(0)} = a_\alpha$. Note that if $m_0(\omega) \neq 0$ for all $\omega \in B^{-1}(U^*)$, then the modified Cohen condition (1.10) is fulfilled with $E = U^*$.

In the sequel, the function $\Phi : \mathbb{R}_+ \to \mathbb{C}$ is related to the refinable function $\varphi : G \to \mathbb{C}$ by the equality $\varphi(x) = \Phi[\lambda(x)]$ for almost every $x \in G$ (obviously, the mapping $\lambda : G \to \mathbb{R}_+$ is invertible almost everywhere). The functions Ψ, $\Psi^{(\nu)}$, and $\widetilde{\Psi}^{(\nu)}$ are defined by analogy with Φ and related to ψ, $\psi^{(\nu)}$, and $\widetilde{\psi}^{(\nu)}$, respectively.

We set

$$b_l^{(\nu)} = m_\nu(B^{-N}\omega_{[l]}), \quad \nu = 0, 1, \ldots, p-1, \quad l = 0, 1, \ldots, p^{n-1} - 1.$$

According to (1.11), we have

$$b_0^{(0)} = 1, \quad |b_l^{(0)}|^2 + |b_{l+p^{n-1}}^{(0)}|^2 + \ldots + |b_{l+(p-1)p^{n-1}}^{(0)}|^2 = 1, \quad l = 1, 2, \ldots, p^{n-1} - 1.$$

By the algorithm given above we need to find the coefficients

$$b_0^{(\nu)}, b_1^{(\nu)}, \ldots, b_{p^n-1}^{(\nu)}, \quad \nu = 1, 2, \ldots, p-1, \tag{2.1}$$

such that

$$\begin{pmatrix} b_l^{(0)} & b_{l+p^{n-1}}^{(0)} & \ldots & b_{l+(p-1)p^{n-1}}^{(0)} \\ b_l^{(1)} & b_{l+p^{n-1}}^{(1)} & \ldots & b_{l+(p-1)p^{n-1}}^{(1)} \\ \ldots & \ldots & \ldots & \ldots \\ b_l^{(p-1)} & b_{l+p^{n-1}}^{(p-1)} & \ldots & b_{l+(p-1)p^{n-1}}^{(p-1)} \end{pmatrix}, \quad l = 0, 1, \ldots, p^{n-1} - 1,$$

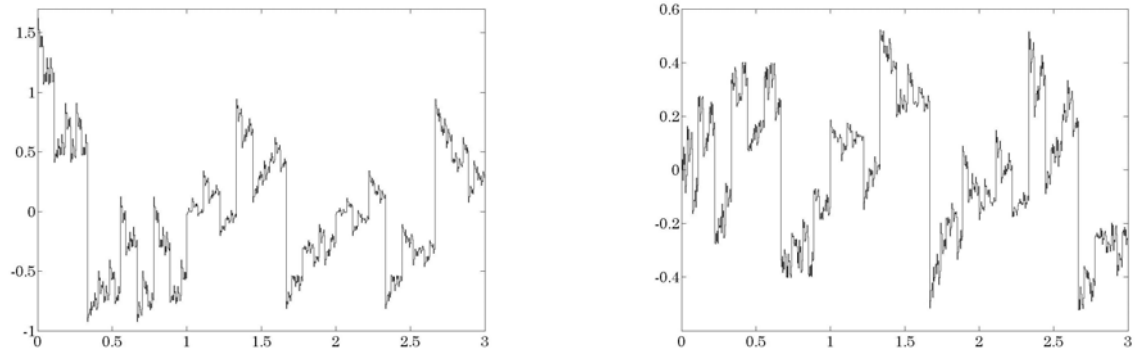

Fig. 2. Re Ψ_1 (left) and Im Ψ_1 (right) from Example 1.

will be unitary matrices. To do so, for each l, $0 \leqslant l \leqslant p^{n-1}-1$, we must solve the following system of equations:

$$\begin{cases} \overline{b_l^{(0)}} b_l^{(\nu)} + \overline{b_{l+p^{n-1}}^{(0)}} b_{l+p^{n-1}}^{(\nu)} + \ldots + \overline{b_{l+(p-1)p^{n-1}}^{(0)}} b_{l+(p-1)p^{n-1}}^{(\nu)} = 0, & \nu = 1, 2, \ldots, p-1, \\ \overline{b_l^{(\nu)}} b_l^{(\mu)} + \overline{b_{l+p^{n-1}}^{(\nu)}} b_{l+p^{n-1}}^{(\mu)} + \ldots + \overline{b_{l+(p-1)p^{n-1}}^{(\nu)}} b_{l+(p-1)p^{n-1}}^{(\mu)} = 0, & \nu, \mu = 1, 2, \ldots, p-1, \end{cases} \tag{2.2}$$

where $l = 0, 1 \ldots, p^{n-1}-1$. Note, that system (2.2) includes $p(p+1)/2-1$ equations with $p(p-1)$ variables.

For $l = 0$, we define

$$R_0 = \frac{|c_1|}{\sqrt{|c_1|^2+|c_2|^2}} \begin{pmatrix} 1 & c_2/c_1 & 0 & \ldots & 0 \\ -\overline{c_2}/\overline{c_1} & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix},$$

$$R_1 = \frac{|c_1^{(1)}|}{\sqrt{|c_1^{(1)}|^2+|c_3|^2}} \begin{pmatrix} 1 & 0 & c_3/c_1^{(1)} & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ -\overline{c_3}/\overline{c_1^{(1)}} & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix},$$

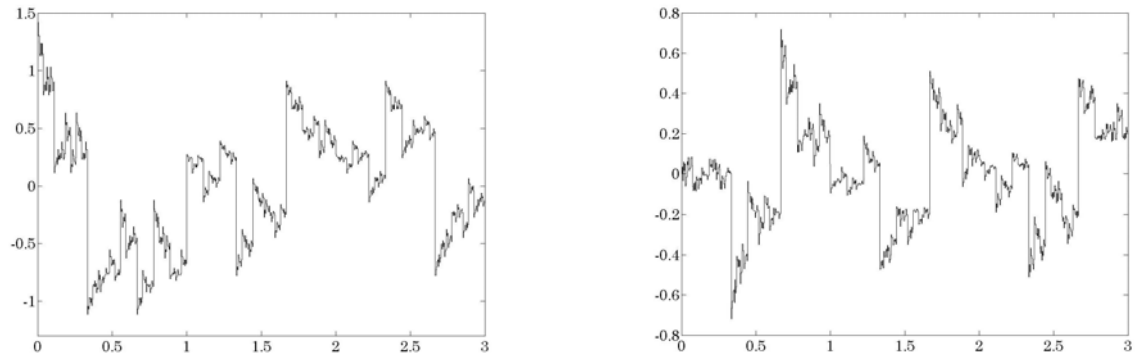

Fig. 3. Re Ψ_2 (left) and Im Ψ_2 (right) from Example 1.

$$R_{p-2} = \frac{|c_1^{(p-2)}|}{\sqrt{|c_1^{(p-2)}|^2 + |c_p|^2}} \begin{pmatrix} 1 & 0 & 0 & \dots & c_p/c_1^{(p-2)} \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\overline{c_p}/\overline{c_1^{(p-2)}} & 0 & 0 & \dots & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where $c_1 = \overline{b_0^{(0)}}$, $c_2 = \overline{b_{p^{n-1}}^{(0)}}$, $\dots$, $c_p = \overline{b_{(p-1)p^{n-1}}^{(0)}}$ and

$$c_1^{(1)} = \frac{|c_1|}{\overline{c_1}}\sqrt{|c_1|^2 + |c_2|^2}, \quad c_1^{(2)} = \frac{|c_1^{(1)}|}{\overline{c_1^{(1)}}}\sqrt{|c_1^{(1)}|^2 + |c_3|^2}, \quad \dots, \quad c_1^{(p-2)} = \frac{|c_1^{(p-3)}|}{\overline{c_1^{(p-3)}}}\sqrt{|c_1^{(p-3)}|^2 + |c_p|^2}.$$

Now, let us consider an unitary transform in $\mathbb{C}^p$ with the matrix $R = R_{p-1}R_{p-2}\dots R_0$. This transform maps the plane

$$c_1 z_1 + c_2 z_2 + \dots + c_p z_p = 0 \tag{2.3}$$

into the plane $z_1' = 0$. Suppose that an orthonormal system $\{w_1, w_2, \dots, w_{p-1}\}$ in $\mathbb{C}^p$ consists of vectors of the form

$$\begin{gathered} w_1 = (0, w_{12}, w_{13}, \dots, w_{1,p})^t, \\ w_2 = (0, w_{22}, w_{23}, \dots, w_{2,p})^t, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ w_{p-1} = (0, w_{p-1,2}, w_{p-1,3}, \dots, w_{p-1,p})^t. \end{gathered}$$

Then a part of coefficients (2.1) can be found from the equalities

$$(b_0^{(\nu)}, b_{p^{n-1}}^{(\nu)}, \dots, b_{(p-1)p^{n-1}}^{(\nu)})^t = R^{-1} w_\nu \quad \nu = 1, \dots, p-1.$$

Applying this algorithm by sequentially for $l = 1, \dots, p^{n-1} - 1$, with

$$c_1 = \overline{b_l^{(0)}}, \; c_2 = \overline{b_{l+p^{n-1}}^{(0)}}, \; \dots, \; c_p = \overline{b_{l+(p-1)p^{n-1}}^{(0)}},$$

we obtain all coefficients (2.1).

EXAMPLE 1 (cf. [8], Example 6). Let $p = 3$, $n = 2$, and the mask m takes the value 1 on $U_{2,0}^*$, vanishes on $U_{2,3}^* \cup U_{2,6}^*$, and be defined on the remaining part of U^* by the equalities

$$m(\omega) = a \text{ for } \omega \in U_{2,1}^*, \qquad m(\omega) = \alpha \text{ for } \omega \in U_{2,2}^*,$$

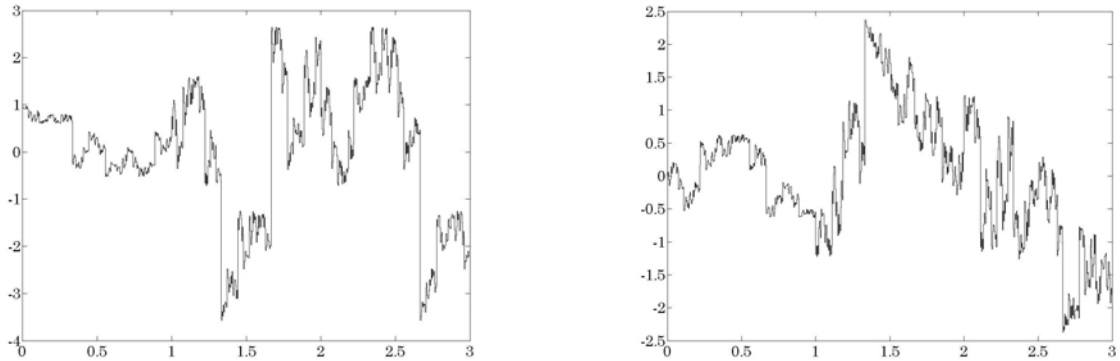

Fig. 4. Re Ψ_1 (left) and Im Ψ_1 (right) from Example 2.

$$m(\omega) = b \text{ for } \omega \in U^*_{2,4}, \qquad m(\omega) = \beta \text{ for } \omega \in U^*_{2,5},$$
$$m(\omega) = c \text{ for } \omega \in U^*_{2,7}, \qquad m(\omega) = \gamma \text{ for } \omega \in U^*_{2,8}.$$

Then for $l = 1$, we have

$$R = \begin{pmatrix} |a| & \frac{|a|\overline{b}}{\overline{a}} & \frac{a\overline{c}}{|a|} \\ -\frac{|a|b}{a\sqrt{1-|c|^2}} & \frac{|a|}{\sqrt{1-|c|^2}} & 0 \\ -\frac{\overline{a}c}{\sqrt{1-|c|^2}} & -\frac{\overline{b}c}{\sqrt{1-|c|^2}} & \sqrt{1-|c|^2} \end{pmatrix}.$$

Let us choose

$$a = 0.900000, \quad \alpha = 0.900000, \quad b = 0.400000, \quad \beta = 0.435890, \quad c = 0.000000, \quad \gamma = 0.383886,$$

and

$$w_1 = \begin{cases} (0,1,0)^t & \text{if} \quad l = 0 \text{ or } l = 2, \\ (0,0,1)^t & \text{if} \quad l = 1, \end{cases} \qquad w_2 = \begin{cases} (0,0,1)^t & \text{if} \quad l = 0 \text{ or } l = 2, \\ (0,1,0)^t & \text{if} \quad l = 1. \end{cases}$$

The corresponding refinable function φ can be defined by (1.9) (see Figure 1). Using our algorithm, we obtain

$$b_0^{(1)} = 0.000000, \quad b_1^{(1)} = 0.000000, \quad b_2^{(1)} = -0.223607, \quad b_3^{(1)} = 1.000000, \quad b_4^{(1)} = 0.000000,$$
$$b_5^{(1)} = 0.974679, \quad b_6^{(1)} = 0.000000, \quad b_7^{(1)} = 1, \quad b_8^{(1)} = 0.000000,$$

$$b_0^{(2)} = 0.000000, \quad b_1^{(2)} = -0.435890, \quad b_2^{(2)} = -0.374166, \quad b_3^{(2)} = 0.000000, \quad b_4^{(2)} = 0.900000,$$
$$b_5^{(2)} = -0.085840, \quad b_6^{(2)} = 1.000000, \quad b_7^{(2)} = 0.000000, \quad b_8^{(2)} = 0.923380.$$

Then

$$\psi^{(\nu)}(x) = 3\sum_{\alpha=0}^{8} a^{(\nu)}\, \varphi(Ax \ominus h_{[\alpha]}), \quad \nu = 1, 2,$$

where $a^{(\nu)}$ are calculated by (1.5) (see Figures 2, 3).

3. AN ALGORITHM FOR CONSTRUCTION OF BIORTHOGONAL WAVELETS

Let φ and $\widetilde{\varphi}$ be with masks

$$m(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W^*_\alpha(\omega)}, \qquad \widetilde{m}(\omega) = \sum_{\alpha=0}^{p^{\widetilde{n}}-1} \widetilde{a}_\alpha \overline{W^*_\alpha(\omega)}, \tag{3.1}$$

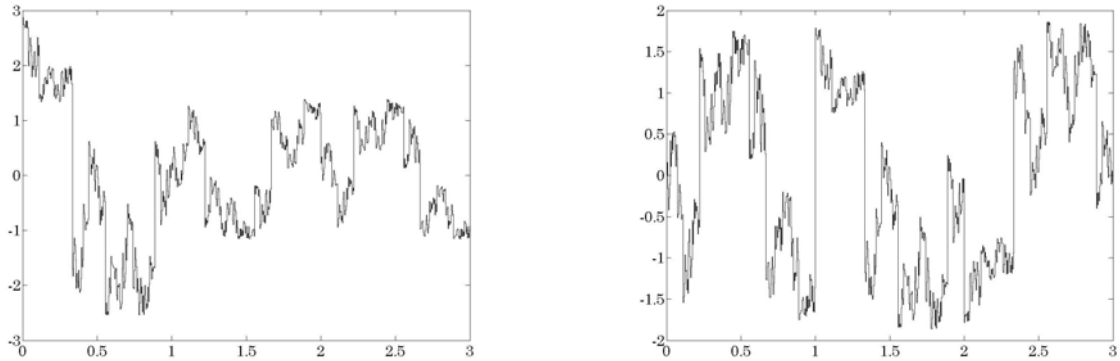

Fig. 5. Re Ψ_2 (left) and Im Ψ_2 (right) from Example 2.

respectively. By analogy with [12], Section 1.2, (see also [9], Proposition 2.2) can be proved the following

PROPOSITION 3.1. *If the systems* $\{\varphi(\cdot \ominus h) \mid h \in H\}$ *and* $\{\widetilde{\varphi}(\cdot \ominus h) \mid h \in H\}$ *are biorthonormal in* $L^2(G)$, *then*

$$\sum_{l=0}^{p-1} m(\omega \oplus \delta_l)\, \overline{\widetilde{m}(\omega \oplus \delta_l)} = 1 \quad \textit{for all} \quad \omega \in G^*. \tag{3.2}$$

Given masks (3.1), we set $m^*(\omega) = m(\omega)\overline{\widetilde{m}(\omega)}$ and $N = \max\{n, \widetilde{n}\}$. Condition (3.2) is then written in the form

$$\sum_{l=0}^{p-1} m^*(\omega \oplus \delta_l) = 1, \qquad \omega \in G^*,$$

and is equivalent to the equalities

$$\sum_{\nu=0}^{p-1} b_{l+\nu p^{N-1}}\, \overline{\widetilde{b}}_{l+\nu p^{N-1}} = 1, \quad 0 \leqslant l \leqslant p^{N-1} - 1, \tag{3.3}$$

where $b_l = m(B^{-N}\omega_{[l]})$, $\widetilde{b}_l = \widetilde{m}(B^{-N}\omega_{[l]})$. Then

$$m^*(\omega) = \sum_{\alpha=0}^{p^n-1} \sum_{\beta=0}^{p^{\widetilde{n}}-1} a_\alpha \overline{\widetilde{a}}_\beta \overline{W^*_{\alpha \oplus_p \beta}(\omega)},$$

where $\oplus_p$ is addition of integers modulo p. Setting $a_\alpha = \widetilde{a}_\beta = 0$ for $\alpha \geqslant p^n$ and $\beta \geqslant p^{\widetilde{n}}$, we obtain

$$m^*(\omega) = \sum_{\alpha=0}^{p^N-1} a^*_\alpha \overline{W^*_\alpha(\omega)}, \qquad a^*_\alpha = \sum_{\gamma=0}^{p^N-1} a_\gamma \overline{\widetilde{a}}_{\gamma \ominus_p \alpha}.$$

Consider the function φ^* defined by

$$\varphi^*(x) = \int_G \varphi(t \oplus x) \overline{\widetilde{\varphi}(t)}\, d\mu(t).$$

The Fourier transform of this function is related to those of φ and $\widetilde{\varphi}$ by $\widehat{\varphi}^*(\omega) = \widehat{\varphi}(\omega)\overline{\widehat{\widetilde{\varphi}}(\omega)}$. Moreover, φ^* is a refinable function satisfying the equation

$$\varphi^*(x) = p \sum_{\alpha=0}^{p^N-1} a^*_\alpha\, \varphi^*(Ax \ominus h_{[\alpha]}), \quad x \in G.$$

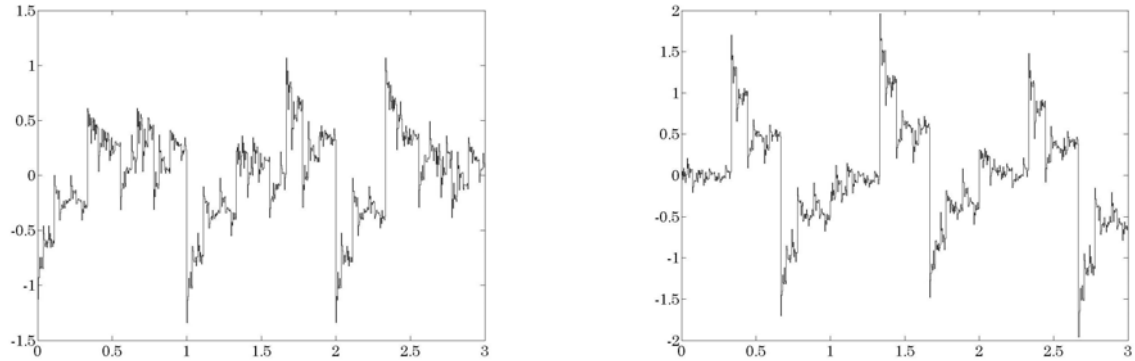

Fig. 6. Re $\widetilde{\Psi}_1$ (left) and Im $\widetilde{\Psi}_1$ (right) from Example 2.

Thus, the polynomial m^* is the mask of the function φ^*. As noted in [9], if one of the masks m, $\widetilde{m}$, m^* has a blocking set, then the systems $\{\varphi(\cdot \ominus h) \mid h \in H\}$ and $\{\widetilde{\varphi}(\cdot \ominus h) \mid h \in H\}$ are not biorthonormal in $L^2(G)$.

According to [9], for the biorthonormal case we have the following:

THEOREM 3.1. *Let φ and $\widetilde{\varphi}$ be refinable functions whose masks $m, \widetilde{m}$ satisfy condition* (3.2) *and $\widehat{\varphi}(\theta) = \widehat{\widetilde{\varphi}}(\theta) = 1$. Then the following conditions are equivalent*:

(a) *the systems $\{\varphi(\cdot \ominus h) \mid h \in H\}$ and $\{\widetilde{\varphi}(\cdot \ominus h) \mid h \in H\}$ are biorthonormal in $L^2(G)$*;

(b) *there exists a compact set E that is congruent U^* to modulo $H^\perp$, contains a neighborhood of zero in G^*, and is such that*

$$\inf_{j\in\mathbb{N}} \inf_{\omega\in E} |m(B^{-j}\omega)| > 0 \quad \textit{and} \quad \inf_{j\in\mathbb{N}} \inf_{\omega\in E} |\widetilde{m}(B^{-j}\omega)| > 0. \tag{3.4}$$

Let $\{V_j\}$ and $\{\widetilde{V}_j\}$ be two MRAs in $L^2(G)$. We say that functions $\psi^{(\nu)} \in V_1$, $\widetilde{\psi}^{(\nu)} \in \widetilde{V}_1$, $\nu = 1, \dots, p-1$, form a *biorthogonal wavelet set* with respect to the pair $\{V_j\}$, $\{\widetilde{V}_j\}$, if $\psi^{(\nu)} \perp \widetilde{V}_0$ and $\widetilde{\psi}^{(\nu)} \perp V_0$ for all $\nu = 1, \dots, p-1$ and

$$(\psi^{(\nu)}(\cdot \oplus h_{[\alpha]}), \widetilde{\psi}^{(\varkappa)}(\cdot \oplus h_{[\beta]})) = \delta_{\nu,\varkappa}\delta_{\alpha,\beta}, \qquad \nu, \varkappa \in \{1, \dots, p-1\}, \quad \alpha, \beta \in \mathbb{Z}_+.$$

As usual, by $\mathcal{M}^*$ we denote the adjoint matrix of $\mathcal{M}$. The identity matrix of order p is denoted by I.

THEOREM 3.2. *Suppose that $\{V_j\}$ and $\{\widetilde{V}_j\}$ are MRAs generated by refinable functions $\varphi, \widetilde{\varphi}$ with masks $m = m_0$ and $\widetilde{m} = \widetilde{m}_0$, respectively, and the systems $\{\varphi(\cdot \ominus h) \mid h \in H\}$ and $\{\widetilde{\varphi}(\cdot \ominus h) \mid h \in H\}$ are biorthonormal. If the matrices*

$$\mathcal{M} = \{m_\nu(\omega \oplus \delta_k)\}_{\nu,k=0}^{p-1}, \quad \widetilde{\mathcal{M}} = \{\widetilde{m}_\nu(\omega \oplus \delta_k)\}_{\nu,k=0}^{p-1},$$

where $m_\nu, \widetilde{m}_\nu \in L^2(U^*)$, for almost every $\omega \in U^*$, satisfy the condition

$$\mathcal{M}\widetilde{\mathcal{M}}^* = I, \tag{3.5}$$

then the functions $\psi^{(\nu)}$ and $\widetilde{\psi}^{(\nu)}$, $\nu = 1, \dots, p-1$, defined by the equalities

$$\widehat{\psi}^{(\nu)}(\omega) = m_\nu(B^{-1}\omega)\widehat{\varphi}(B^{-1}\omega), \qquad \widehat{\widetilde{\psi}^{(\nu)}}(\omega) = \widetilde{m}_\nu(B^{-1}\omega)\widehat{\widetilde{\varphi}}(B^{-1}\omega),$$

form a biorthogonal wavelet set with respect to the pair $\{V_j\}$, $\{\widetilde{V}_j\}$.

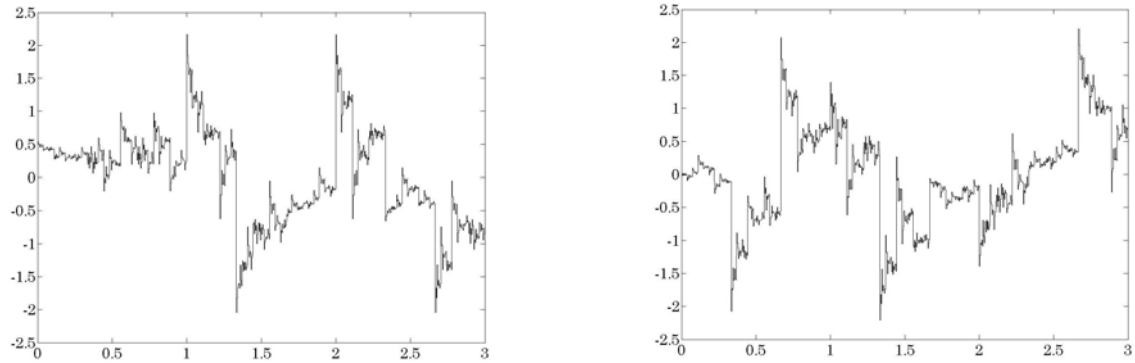

Fig. 7. Re $\widetilde{\Psi}_2$ (left) and Im $\widetilde{\Psi}_2$ (right) from Example 2.

Analog of this theorem for the space $L^2(\mathbb{R}^d)$ is proved in [13], Ch. 2.

Thus, we have the following *procedure to construct a biorthogonal wavelet set in* $L^2(G)$:

1. Choose numbers b_s and $\widetilde{b}_s$ with $0 \leqslant s \leqslant p^N - 1$, so that (3.3) is true.

2. Compute $a_\alpha^{(0)} = a_\alpha$ and $\widetilde{a}_\alpha^{(0)} = \widetilde{a}_\alpha$ with $0 \leqslant \alpha \leqslant p^N - 1$ by (1.5) and verify that the masks

$$m_0(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha^{(0)} \overline{W_\alpha^*(\omega)}, \qquad \widetilde{m}_0(\omega) = \sum_{\alpha=0}^{p^{\widetilde{n}}-1} \widetilde{a}_\alpha^{(0)} \overline{W_\alpha^*(\omega)}.$$

satisfy the condition (b) of Theorem 3.1.

3. Find

$$m_\nu(\omega) = \sum_{\alpha\in\mathbb{Z}_+} a_\alpha^{(\nu)} \overline{W_\alpha^*(\omega)}, \quad \widetilde{m}_\nu(\omega) = \sum_{\alpha\in\mathbb{Z}_+} \widetilde{a}_\alpha^{(\nu)} \overline{W_\alpha^*(\omega)}, \quad 1 \leqslant \nu \leqslant p-1,$$

such that (3.5) is valid for almost every $\omega \in U^*$.

4. Define $\psi^{(\nu)}$ and $\widetilde{\psi}^{(\nu)}$ by the equalities

$$\psi^{(\nu)}(x) = p \sum_{\alpha\in\mathbb{Z}_+} a_\alpha^{(\nu)} \varphi(Ax \ominus h_{[\alpha]}), \quad \widetilde{\psi}^{(\nu)}(x) = p \sum_{\alpha\in\mathbb{Z}_+} \widetilde{a}_\alpha^{(\nu)} \varphi(Ax \ominus h_{[\alpha]}), \qquad 1 \leqslant \nu \leqslant p-1.$$

We note that expansions of the form (1.9) for φ and $\widetilde{\varphi}$ are derived from Proposition 1.2 when $\widehat{\rho}[m] < 1$ and $\widehat{\rho}[\widetilde{m}] < 1$ (this condition can be verified numerically for specific parameter values).

Let us set

$$b_l^{(\nu)} = m_\nu(B^{-N}\omega_{[l]}), \quad \widetilde{b}_l^{(\nu)} = \widetilde{m}_\nu(B^{-N}\omega_{[l]}), \quad \nu = 0, 1, \ldots, p-1, \quad l = 0, 1, \ldots, p^{N-1} - 1.$$

According to (3.3), we have

$$\overline{b_l^{(0)}}\,\widetilde{b}_l^{(0)} + \overline{b_{l+p^{N-1}}^{(0)}}\,\widetilde{b}_{l+p^{N-1}}^{(0)} + \ldots + \overline{b_{l+(p-1)p^{N-1}}^{(0)}}\,\widetilde{b}_{l+(p-1)p^{N-1}}^{(0)} = 1, \quad l = 0, 1, \ldots, p^{N-1} - 1.$$

In order to apply Theorem 3.1, we must find the coefficients

$$b_l^{(\nu)}, b_{l+p^{N-1}}^{(\nu)}, \ldots, b_{l+(p-1)p^{N-1}}^{(\nu)}, \quad \widetilde{b}_l^{(\nu)}, \widetilde{b}_{l+p^{N-1}}^{(\nu)}, \ldots, \widetilde{b}_{l+(p-1)p^{N-1}}^{(\nu)}, \tag{3.6}$$

where $\nu = 1, 2, \ldots p-1$, $l = 0, 1, \ldots, p^{N-1} - 1$, such that the matrices

$$\mathcal{M}_l = \begin{pmatrix} b_l^{(0)} & b_{l+p^{N-1}}^{(0)} & \ldots & b_{l+(p-1)p^{N-1}}^{(0)} \\ b_l^{(1)} & b_{l+p^{N-1}}^{(1)} & \ldots & b_{l+(p-1)p^{N-1}}^{(1)} \\ \ldots & \ldots & \ldots & \ldots \\ b_l^{(p-1)} & b_{l+p^{N-1}}^{(p-1)} & \ldots & b_{l+(p-1)p^{N-1}}^{(p-1)} \end{pmatrix}, \quad \widetilde{\mathcal{M}}_l = \begin{pmatrix} \widetilde{b}_l^{(0)} & \widetilde{b}_{l+p^{N-1}}^{(0)} & \ldots & \widetilde{b}_{l+(p-1)p^{N-1}}^{(0)} \\ \widetilde{b}_l^{(1)} & \widetilde{b}_{l+p^{N-1}}^{(1)} & \ldots & \widetilde{b}_{l+(p-1)p^{N-1}}^{(1)} \\ \ldots & \ldots & \ldots & \ldots \\ \widetilde{b}_l^{(p-1)} & \widetilde{b}_{l+p^{N-1}}^{(p-1)} & \ldots & \widetilde{b}_{l+(p-1)p^{N-1}}^{(p-1)} \end{pmatrix},$$

satisfy the condition

$$\mathcal{M}_l\widetilde{\mathcal{M}}_l^* = I, \quad l = 0,\, 1,\, \ldots,\, p^{N-1}-1.$$

To do so, we solve for each l, $0 \leqslant l \leqslant p^{n-1}-1$, the following system of equations

$$\begin{cases} \overline{b_l^{(0)}}\,\widetilde{b}_l^{(\nu)} + \overline{b_{l+p^{N-1}}^{(0)}}\,\widetilde{b}_{l+p^{N-1}}^{(\nu)} + \ldots + \overline{b_{l+(p-1)p^{N-1}}^{(0)}}\,\widetilde{b}_{l+(p-1)p^{N-1}}^{(\nu)} = 0, \quad \nu = 1,\, 2\, \ldots,\, p-1, \\ \overline{\widetilde{b}_l^{(0)}}\,b_l^{(\nu)} + \overline{\widetilde{b}_{l+p^{N-1}}^{(0)}}\,b_{l+p^{N-1}}^{(\nu)} + \ldots + \overline{\widetilde{b}_{l+(p-1)p^{N-1}}^{(0)}}\,b_{l+(p-1)p^{N-1}}^{(\nu)} = 0, \quad \nu = 1,\, 2\, \ldots,\, p-1, \\ \overline{\widetilde{b}_l^{(\nu)}}\,b_l^{(\mu)} + \overline{\widetilde{b}_{l+p^{N-1}}^{(\nu)}}\,b_{l+p^{N-1}}^{(\mu)} + \ldots + \overline{\widetilde{b}_{l+(p-1)p^{N-1}}^{(\nu)}}\,b_{l+(p-1)p^{N-1}}^{(\mu)} = \delta_{\nu,\mu}, \quad \nu,\mu = 1,\, 2\, \ldots,\, p-1. \end{cases} \tag{3.7}$$

Note, that the system (3.7) consists of p^2-1 equations with $2p(p-1)$ variables.

For $l = 0$, we put

$$c_1 = \overline{b_0^{(0)}},\, c_2 = \overline{b_{p^{N-1}}^{(0)}},\, \ldots,\, c_p = \overline{b_{(p-1)p^{N-1}}^{(0)}}, \quad \widetilde{c}_1 = \overline{\widetilde{b}_0^{(0)}},\, \widetilde{c}_2 = \overline{\widetilde{b}_{p^{N-1}}^{(0)}},\, \ldots,\, \widetilde{c}_p = \overline{\widetilde{b}_{(p-1)p^{N-1}}^{(0)}},$$

and consider the following planes:

$$c_1z_1 + c_2z_2 + \cdots + c_pz_p = 0, \tag{3.8}$$

$$\widetilde{c}_1z_1 + \widetilde{c}_2z_2 + \cdots + \widetilde{c}_pz_p = 0. \tag{3.9}$$

As in orthogonal case, using an unitary transform, we can map the planes (3.8) and (3.9) into the planes

$$c_1'z_1' + c_2'z_2' + \cdots + c_p'z_p' = 0, \tag{3.10}$$

$$z_1' = 0, \tag{3.11}$$

respectively. Now, we choose vectors $w_i = (0, w_{i2}, w_{i3}, \ldots, w_{ip})^t$, $i = 1,\, 2,\, \ldots,\, p-1$, such that

$$W = \begin{pmatrix} \overline{w_{12}} & \overline{w_{13}} & \ldots & \overline{w_{1,p}} \\ \overline{w_{22}} & \overline{w_{23}} & \ldots & \overline{w_{2,p}} \\ \ldots & \ldots & \ldots & \ldots \\ \overline{w_{p-1,2}} & \overline{w_{p-1,3}} & \ldots & \overline{w_{p-1,p}} \end{pmatrix}$$

is a nonsingular matrix. Further, let us find the vectors $\widetilde{w}_i = (\widetilde{w}_{i1}, \widetilde{w}_{i2}, \ldots, \widetilde{w}_{ip})^t$, $i = 1, 2, \ldots, p-1$, such that the following relations hold

$$\langle w_i, \widetilde{w}_j\rangle = \delta_{i,j}, \quad i,j = 1,\, 2,\, \ldots,\, p-1. \tag{3.12}$$

To find $\widetilde{w}_{i,k}$ with $k \geqslant 2$ we solve the following systems of the linear equations

$$W\begin{pmatrix} \widetilde{w}_{1,2} \\ \widetilde{w}_{1,3} \\ \ldots \\ \widetilde{w}_{1,p} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix},\; W\begin{pmatrix} \widetilde{w}_{2,2} \\ \widetilde{w}_{2,3} \\ \ldots \\ \widetilde{w}_{2,p} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \ldots \\ 0 \end{pmatrix},\; \ldots,\; W\begin{pmatrix} \widetilde{w}_{p-1,2} \\ \widetilde{w}_{p-1,3} \\ \ldots \\ \widetilde{w}_{p-1,p} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \ldots \\ 1 \end{pmatrix}.$$

The first components $\widetilde{w}_{i1}$ can be found from the condition that vectors $\widetilde{w}_i$ satisfy (3.9):

$$\widetilde{w}_{i1} = -\frac{1}{c_1'}(c_2'\widetilde{w}_{i2} + c_3'\widetilde{w}_{i3} + \ldots + c_p'\widetilde{w}_{i,p}), \quad i = 1,\, 2,\, \ldots,\, p-1,$$

where $c_1', \ldots, c_p'$ are taken from (3.10). Therefore, we have obtained vectors $\widetilde{w}_i$ satisfying (3.10) and (3.12). After that we return to the original coordinates $z_1, z_2, \ldots, z_p$. Applying this algorithm sequentially for $l = 1, \ldots, p^{n-1}-1$ and setting

$$c_1 = \overline{b_l^{(0)}},\, c_2 = \overline{b_{l+p^{N-1}}^{(0)}},\, \ldots,\, c_p = \overline{b_{l+(p-1)p^{N-1}}^{(0)}}, \quad \widetilde{c}_1 = \overline{\widetilde{b}_l^{(0)}},\, \widetilde{c}_2 = \overline{\widetilde{b}_{l+p^{N-1}}^{(0)}},\, \ldots,\, \widetilde{c}_p = \overline{\widetilde{b}_{l+(p-1)p^{N-1}}^{(0)}},$$

we obtain all coefficients (3.6).

EXAMPLE 2. Let $p=3$, $n=\widetilde{n}=2$, and the masks m, $\widetilde{m}$ take the value 1 on $U^*_{2,0}$, vanish on $U^*_{2,3}\cup U^*_{2,6}$, and be defined on the remaining part of the U^* by the equalities

$$m(\omega)=a \text{ and } \widetilde{m}(\omega)=\widetilde{a} \text{ for } \omega\in U^*_{2,1}, \qquad m(\omega)=\alpha \text{ and } \widetilde{m}(\omega)=\widetilde{\alpha} \text{ for } \omega\in U^*_{2,2},$$

$$m(\omega)=b \text{ and } \widetilde{m}(\omega)=\widetilde{b} \text{ for } \omega\in U^*_{2,4}, \qquad m(\omega)=\beta \text{ and } \widetilde{m}(\omega)=\widetilde{\beta} \text{ for } \omega\in U^*_{2,5},$$

$$m(\omega)=c \text{ and } \widetilde{m}(\omega)=\widetilde{c} \text{ for } \omega\in U^*_{2,7}, \qquad m(\omega)=\gamma \text{ and } \widetilde{m}(\omega)=\widetilde{\gamma} \text{ for } \omega\in U^*_{2,8},$$

where the parameters satisfy the condition

$$a\overline{\widetilde{a}}+b\overline{\widetilde{b}}+c\overline{\widetilde{c}}=\alpha\overline{\widetilde{\alpha}}+\beta\overline{\widetilde{\beta}}+\gamma\overline{\widetilde{\gamma}}=1.$$

Then, in the cases $a\overline{\widetilde{a}}=\alpha\overline{\widetilde{\alpha}}=0$, $a\overline{\widetilde{a}}=c\overline{\widetilde{c}}=0$ and $\alpha\overline{\widetilde{\alpha}}=\beta\overline{\widetilde{\beta}}=0$, blocking sets for the mask m^* are $U^*_{1,1}\cup U^*_{1,2}$, $U^*_{1,1}$ and $U^*_{1,2}$, respectively. Condition (3.4) holds in the following three cases:

1) $a\widetilde{a}\neq 0$, $\alpha\widetilde{\alpha}\neq 0$ and $E=U^*$;

2) $a\widetilde{a}\neq 0$, $\beta\widetilde{\beta}\neq 0$ and $E=U^*_{1,0}\cup U^*_{1,1}\cup U^*_{1,5}$;

3) $c\widetilde{c}\neq 0$, $\alpha\widetilde{\alpha}\neq 0$ and $E=U^*_{1,0}\cup U^*_{1,2}\cup U^*_{1,7}$.

Note also that the matrix

$$\begin{pmatrix} \frac{|\widetilde{a}|}{\sqrt{|\widetilde{a}|^2+|\widetilde{b}|^2+|\widetilde{c}|^2}} & \frac{|\widetilde{a}|\overline{\widetilde{b}}}{\overline{\widetilde{a}}\sqrt{|\widetilde{a}|^2+|\widetilde{b}|^2+|\widetilde{c}|^2}} & \frac{\widetilde{a}\overline{\widetilde{c}}}{|\widetilde{a}|\sqrt{|\widetilde{a}|^2+|\widetilde{b}|^2+|\widetilde{c}|^2}} \\ -\frac{|\widetilde{a}|\widetilde{b}}{\widetilde{a}\sqrt{|\widetilde{a}|^2+|\widetilde{b}|^2}} & \frac{|\widetilde{a}|}{\sqrt{|\widetilde{a}|^2+|\widetilde{b}|^2}} & 0 \\ -\frac{\overline{\widetilde{a}}\widetilde{c}}{\sqrt{|\widetilde{a}|^2+|\widetilde{b}|^2}\sqrt{|\widetilde{a}|^2+|\widetilde{b}|^2+|\widetilde{c}|^2}} & -\frac{\overline{\widetilde{b}}\widetilde{c}}{\sqrt{|\widetilde{a}|^2+|\widetilde{b}|^2}\sqrt{|\widetilde{a}|^2+|\widetilde{b}|^2+|\widetilde{c}|^2}} & \frac{\sqrt{|\widetilde{a}|^2+|\widetilde{b}|^2}}{\sqrt{|\widetilde{a}|^2+|\widetilde{b}|^2+|\widetilde{c}|^2}} \end{pmatrix}.$$

maps the plane $\widetilde{a}z_1+\widetilde{b}z_2+\widetilde{c}z_3=0$ into the plane $z_1'=0$. Let us choose

$$a=0.900000,\quad \alpha=0.900000,\quad b=-0.295272,\quad \beta=0.478760,\quad c=0.403503,\quad \gamma=0.144181,$$

$$\widetilde{a}=0.900000,\quad \widetilde{\alpha}=0.900000,\quad \widetilde{b}=0.037839,\quad \widetilde{\beta}=0.270151,\quad \widetilde{c}=0.498566,\quad \widetilde{\gamma}=0.420735.$$

The corresponding refinable functions φ and $\widetilde{\varphi}$ were given in [9], Example 3. Let

$$w_1=\begin{cases}(0,2,0)^t & \text{if } l=0,\\ (0,1,-3)^t & \text{if } l=1,\\ (0,-1,3)^t & \text{if } l=2,\end{cases} \qquad w_2=\begin{cases}(0,2,1/2)^t & \text{if } l=0,\\ (0,4,1)^t & \text{if } l=1,\\ (0,-1/2,0)^t & \text{if } l=2.\end{cases}$$

Using our algorithm, we obtain

$$b_0^{(1)}=0.000000,\quad b_1^{(1)}=1.409462,\quad b_2^{(1)}=-0.886709,\quad b_3^{(1)}=2.000000,\quad b_4^{(1)}=1.060142,$$
$$b_5^{(1)}=-1.310240,\quad b_6^{(1)}=0.000000,\quad b_7^{(1)}=-2.624789,\quad b_8^{(1)}=2.738068,$$

$$b_0^{(2)}=0.000000,\quad b_1^{(2)}=-0.651848,\quad b_2^{(2)}=0.143748,\quad b_3^{(2)}=2.000000,\quad b_4^{(2)}=3.976128,$$
$$b_5^{(2)}=-0.478891,\quad b_6^{(2)}=0.500000,\quad b_7^{(2)}=0.874930,\quad b_8^{(2)}=0.000000,$$

$$\widetilde{b}_0^{(1)}=0.000000,\quad \widetilde{b}_1^{(1)}=0.147522,\quad \widetilde{b}_2^{(1)}=-0.047392,\widetilde{b}_3^{(1)}=0.500000,\quad \widetilde{b}_4^{(1)}=0.083193,$$
$$\widetilde{b}_5^{(1)}=-0.014226,\quad \widetilde{b}_6^{(1)}=-2.000000,\quad \widetilde{b}_7^{(1)}=-0.268165,\quad \widetilde{b}_8^{(1)}=0.343066,$$

$$\widetilde{b}_0^{(2)}=0.000000,\quad \widetilde{b}_1^{(2)}=0.027502,\quad \widetilde{b}_2^{(2)}=1.029419,\quad \widetilde{b}_3^{(2)}=0.000000,\quad \widetilde{b}_4^{(2)}=0.232129,$$
$$\widetilde{b}_5^{(2)}=-1.779160,\quad \widetilde{b}_6^{(2)}=2.000000,\quad \widetilde{b}_7^{(2)}=0.108524,\quad \widetilde{b}_8^{(2)}=-0.518004.$$

The real and imaginary parts of $\Psi_1, \Psi_2, \widetilde{\Psi}_1, \widetilde{\Psi}_2$ are plotted in Figures 4, 5, 6, 7.

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