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Estimates of the Smoothness of Dyadic Orthogonal Wavelets of Daubechies Type

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Abstract—Suppose that $\omega(\varphi, \cdot)$ is the dyadic modulus of continuity of a compactly supported function φ in $L^2(\mathbb{R}_+)$ satisfying a scaling equation with 2^n coefficients. Denote by α_φ the supremum for values of $\alpha > 0$ such that the inequality $\omega(\varphi, 2^{-j}) \le C2^{-\alpha j}$ holds for all $j \in \mathbb{N}$. For the cases $n = 3$ and $n = 4$, we study the scaling functions φ generating multiresolution analyses in $L^2(\mathbb{R}_+)$ and the exact values of α_φ are calculated for these functions. It is noted that the smoothness of the dyadic orthogonal wavelet in $L^2(\mathbb{R}_+)$ corresponding to the scaling function φ coincides with α_φ.

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1. INTRODUCTION

It is well known that the Daubechies scaling function of order N is a solution of the functional equation

$$\varphi(x) = \sqrt{2} \sum_{k=0}^{2N-1} h_k \varphi(2x - k), \qquad x \in \mathbb{R}, \tag{1}$$

and possesses the following properties:

1) $\operatorname{supp} \varphi = [0, 2N - 1]$;

2) the system $\{\varphi(\,\cdot\, - k) : k \in \mathbb{Z}\}$ is orthonormal in $L^2(\mathbb{R})$;

3) φ generates a multiresolution analysis in $L^2(\mathbb{R})$.

For $N = 1$, the Daubechies construction yields the Haar function: $\varphi = \chi_{[0,1)}$ (in this case, $h_0 = h_1 = 1/\sqrt{2}$). For $2 \le N \le 10$, the coefficients of Eq. (1) are given in Daubechies' book (see [1, Sec. 6.4]). For $N = 2$, the solution of Eq. (1) is continuous on $\mathbb{R}$ and satisfies the Lipschitz condition

$$|\varphi(t) - \varphi(x)| \le C|t - x|^\alpha, \qquad t, x \in \mathbb{R},$$

with exponent $\alpha \approx 0.5500$. The exact value of the exponent α (and the corresponding quantities for $N = 3$ and $N = 4$) were obtained earlier by the matrix method (see [1, Sec. 7.2], [2, Sec. 7.3]). For Daubechies scaling functions of order $N \ge 5$, the exact values of the exponents of smoothness are not known to the authors.

The formula expressing the smoothness of the scaling function φ in terms of the joint spectral radius of some matrices generated by the coefficients of Eq. (1), was first given in the papers by Daubechies and

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Lagarias [3] (see also [4]); below we shall use the "dyadic" analog of this formula. Let us recall that the joint spectral radius of two complex matrices A_0 and A_1 of dimensions $N \times N$ is defined by the formula

$$\widehat{\rho}(A_0, A_1) := \lim_{k\to\infty} \max\{\|A_{d_1} A_{d_2} \cdots A_{d_k}\|^{1/k} : d_j \in \{0,1\},\ 1 \le j \le k\},$$

where $\|\cdot\|$ is an arbitrary norm on $\mathbb{C}^{N\times N}$. Obviously, if $A_0 = A_1$, then the quantity $\widehat{\rho}(A_0, A_1)$ coincides with the spectral radius $\rho(A_0)$.

Suppose that $\mathbb{R}_+ = [0, +\infty)$ is the positive half-line, $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$ and $\mathbb{N}$ is the set of natural numbers. For each $x \in \mathbb{R}_+$ and any $j \in \mathbb{N}$, we define $x_{-j}, x_j \in \{0, 1\}$ from the binary expansion

$$x = \sum_{j=1}^{\infty} x_{-j} 2^{j-1} + \sum_{j=1}^{\infty} x_j 2^{-j}$$

(in the case of a binary rational number x, we chose an expansion with a finite number of nonzero terms). The binary addition on $\mathbb{R}_+$ is defined by the formula

$$x \oplus y = \sum_{j=1}^{\infty} |x_{-j} - y_{-j}| 2^{j-1} + \sum_{j=1}^{\infty} |x_j - y_j| 2^{-j}, \qquad x, y \in \mathbb{R}_+;$$

it plays a key role in the theory of Walsh series and transforms (see [5]–[7]). Thus, for example, if the sum $x \oplus y$ is binary irrational, then the classical Walsh functions $\{w_k\}$ satisfy the following relation:

$$w_k(x \oplus y) = w_k(x) w_k(y).$$

The recent paper [8] furnishes necessary and sufficient conditions under which the dyadic scaling equation

$$\varphi(x) = \sum_{k=0}^{2^n-1} c_k \varphi(2x \oplus k), \qquad x \in \mathbb{R}_+, \tag{2}$$

has a solution $\varphi \in L^2(\mathbb{R}_+)$ with the following properties:

1) $\operatorname{supp}\varphi = [0, 2^{n-1}]$;

2) the system $\{\varphi(\,\cdot\, \oplus k) : k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$;

3) φ generates a multiresolution analysis in $L^2(\mathbb{R}_+)$.

In particular, if $n = 1$ and $c_0 = c_1 = 1$, then the solution of Eq. (2) is the Haar function (see also examples in [9], [10]). In the present paper, we calculate the exact values of the exponents of smoothness for the solutions of Eq. (2) in the cases $n = 3$ and $n = 4$. In the proofs of lower bounds, we essentially use expansions of dyadic scaling functions in Walsh series (this is the main distinction of our method from the previously used methods for estimating the smoothness of wavelets). Note that an algorithm for the expansion of scaling functions in lacunary Walsh series was obtained by the second author in [11], and it follows from results given in [8] that this algorithm can be applied to any compactly supported dyadic scaling function in the space $L^2(\mathbb{R}_+)$. The coefficients of Eq. (2) are assumed real, although all the values (given below) of the joint spectral radius of the matrices generated by these coefficients remain valid in the complex case as well.

The Walsh polynomial

$$m(\omega) = \frac{1}{2} \sum_{k=0}^{2^n-1} c_k w_k(\omega), \qquad \omega \in \mathbb{R}_+,$$

is called the *mask* of the scaling equation (2). It is well known that the mask $m(\omega)$ is constant on binary intervals of rank n. Moreover, if $b_l = m(\omega)$ for

$$\omega \in [l2^{-n}, (l+1)2^{-n}), \qquad 0 \le l \le 2^n - 1,$$

then

$$c_k = \frac{1}{2^{n-1}} \sum_{l=0}^{2^n-1} b_l w_l(k2^{-n}), \qquad 0 \le k \le 2^n - 1. \tag{3}$$

The dyadic modulus of continuity of the scaling function φ satisfying Eq. (2) is defined by the equality

$$\omega(\varphi, \delta) := \sup\{|\varphi(x \oplus y) - \varphi(x)| : x, y \in [0, 2^{n-1}),\ 0 \le y < \delta\}, \qquad \delta > 0.$$

If φ satisfies

$$\omega(\varphi, 2^{-j}) \le C2^{-\alpha j}, \qquad j \in \mathbb{N},$$

for some $\alpha > 0$, then there exists [6, Sec. 5.1] a constant $C(\varphi, \alpha)$ such that

$$\omega(\varphi, \delta) \le C(\varphi, \alpha)\delta^{\alpha}. \tag{4}$$

Denote by α_φ the supremum for the set of all values of $\alpha > 0$ for which inequality (4) holds.

The definition of a multiresolution analysis in $L^2(\mathbb{R}_+)$ was given in [8] and is quite similar to the corresponding definition for the space $L^2(\mathbb{R})$ (see, for example, [1], [2]). The solution φ of Eq. (2) is said to *generate a multiresolution analysis in* $L^2(\mathbb{R}_+)$ if, first, the system $\{\varphi(\cdot \oplus k) : k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$ and, second, the family of closed subspaces

$$V_j = \operatorname{clos}_{L^2(\mathbb{R}_+)} \operatorname{span}\{\varphi(2^j \cdot \oplus k) : k \in \mathbb{Z}_+\}, \qquad j \in \mathbb{Z},$$

is a multiresolution analysis in $L^2(\mathbb{R}_+)$.

It was shown in [8] that the equalities

$$b_0 = 1, \qquad |b_l|^2 + |b_{l+2^{n-1}}|^2 = 1, \quad 0 \le l \le 2^{n-1} - 1,$$

are necessary for the orthonormality of the system $\{\varphi(\cdot \oplus k) : k \in \mathbb{Z}_+\}$ in $L^2(\mathbb{R}_+)$. For $n = 2$, we can take

$$b_0 = 1, \quad b_1 = \sqrt{1 - |b|^2}, \quad b_2 = 0, \quad b_3 = b, \qquad \text{where} \quad 0 < |b| < 1,$$

and then $\alpha_\varphi = \log_2(1/|b|)$ (see [8, Remark 3], [11, Example 4.3]).

In the general case, we let $N = 2^{n-1}$ and define the $N \times N$ matrices T_0, T_1 by the formulas

$$(T_0)_{ij} = c_{2(i-1)\oplus(j-1)}, \qquad (T_1)_{ij} = c_{(2i-1)\oplus(j-1)}, \quad 1 \le i, j \le N, \tag{5}$$

where the c_k are the coefficients of Eq. (2).

Proposition 1. *Suppose that the compactly supported solution φ of Eq. (2) generates a multiresolution analysis in $L^2(\mathbb{R}_+)$, and suppose that $\widehat{\rho} = \widehat{\rho}(L_0, L_1)$ is the joint spectral radius of the linear operators L_0, L_1 given on $\mathbb{R}^N$ by the matrices T_0, T_1 and restricted to the subspace*

$$E_1 := \{u = (u_1, \dots, u_N)^t : u_1 + \dots + u_N = 0\}.$$

Then $\alpha_\varphi = -\log_2 \widehat{\rho}$.

Note that, for $n = 3$, the basis of the space E_1 is constituted by the vectors

$$e_1 = (1, -1, 1, -1)^t, \qquad e_2 = (1, -1, -1, 1)^t, \qquad e_3 = (1, 1, -1, -1)^t \tag{6}$$

and the matrices (5) are of the form

$$T_0 = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 \\ c_2 & c_3 & c_0 & c_1 \\ c_4 & c_5 & c_6 & c_7 \\ c_6 & c_7 & c_4 & c_5 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} c_1 & c_0 & c_3 & c_2 \\ c_3 & c_2 & c_1 & c_0 \\ c_5 & c_4 & c_7 & c_6 \\ c_7 & c_6 & c_5 & c_4 \end{pmatrix}. \tag{7}$$

Proposition 2. *For any $d \times d$ matrices A_0, A_1 and for any $q \geq 0$, the following conditions are equivalent*:

1) *there exist constants $C > 0$, $p \geq 0$ such that, for all $m \in \mathbb{N}$,*

$$\max\{\|A_{d_1} \cdots A_{d_m}\| : d_j \in \{0,1\},\ 1 \leq j \leq m\} \leq C m^p q^m;$$

2) $\widehat{\rho}(A_0, A_1) \leq q$.

Moreover, if any one of these properties holds, then $p \leq d - 1$ can always be taken, but if the matrices are irreducible (*have no nontrivial real eigensubspaces in common*), *then $p = 0$.*

The proof of Proposition 1 is quite similar to the proofs of the corresponding statements in [2] and [12] (see also [8, Remark 3]), while Proposition 2 was proved in [12].

For an arbitrary $N \times N$ matrix T, we set

$$\|T|_{E_1}\| := \sup\left\{\frac{\|Tu\|}{\|u\|} : u \in E_1,\ u \neq 0\right\},$$

where $\|u\|$ is the Euclidean norm of the vector u and E_1 is the subspace from Proposition 1. It follows from Propositions 1 and 2 that if the compactly supported solution φ of Eq. (2) generates a multiresolution analysis in $L^2(\mathbb{R}_+)$ and, for all $m \in \mathbb{N}$, the following inequality holds:

$$\max\{\|T_{d_1} T_{d_2} \cdots T_{d_m}|_{E_1}\| : d_j \in \{0,1\},\ 1 \leq j \leq m\} \leq C m^p q^m,$$

where $0 \leq q < 1$, $p \geq 0$, then $\widehat{\rho}(L_0, L_1) \leq q$.

In the course of computer experiments whose results are given below, the joint spectral radius of the matrices was calculated from its definition using the Matlab 7.0 system; in addition, the method of estimating the quantity $\widehat{\rho}$ with the help of Kronecker products (see [2, p. 309], [13]) was used. Also note that the smoothness of the dyadic orthogonal wavelet ψ in $L^2(\mathbb{R}_+)$ corresponding to the scaling function φ coincides with α_φ (for the details, see [8, Sec. 4]).

The main results of the present paper were announced in the abstract of our paper [14] at the Fifth International Symposium "Fourier Series and Their Applications."

2. ESTIMATES OF α_φ FOR $n = 3$

Let us define the coefficients of the scaling equation (2) using formulas (3), where the parameters are taken as

$$b_0 = 1, \quad b_1 = a, \quad b_2 = b, \quad b_3 = c, \quad b_4 = 0, \quad b_5 = \alpha, \quad b_6 = \beta, \quad b_7 = \gamma;$$

here

$$|a|^2 + |\alpha|^2 = |b|^2 + |\beta|^2 = |c|^2 + |\gamma|^2 = 1$$

(see [8, Example 4]). In this case, the matrices of the linear operators L_0 and L_1 given on $\mathbb{R}^4$ by the matrices (7) and restricted to the subspace E_1 with basis (6) are of the form

$$A_0 = \begin{pmatrix} 0 & \beta & b \\ 0 & \gamma & c \\ \alpha & 0 & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & -\beta & b \\ 0 & -\gamma & c \\ -\alpha & 0 & 0 \end{pmatrix} \tag{8}$$

and, further, $\widehat{\rho}(L_0, L_1) = \widehat{\rho}(A_0, A_1)$. Using computer methods, we obtain the following results.

A. The equality $\widehat{\rho}(A_0, A_1) = \rho(A_0)$ holds in the cases:

1) $0 \leq b \leq 1, \quad 0 < c \leq 1, \quad 0 \leq \alpha < 1, \quad 0 \leq \beta\gamma < 1;$

2) $-1 \leq b \leq 0, \quad -1 \leq c < 0, \quad -1 < \alpha \leq 0, \quad 0 \leq \beta\gamma < 1;$

$$3)\quad 0 \le b \le 1,\quad -1 \le c < 0,\quad 0 \le \alpha < 1,\quad -1 < \beta\gamma \le 0;$$
$$4)\quad -1 \le b \le 0,\quad 0 < c \le 1,\quad -1 < \alpha \le 0,\quad -1 < \beta\gamma \le 0.$$

B. The equality $\widehat{\rho}(A_0, A_1) = \rho(A_1)$ holds in the cases:

$$1)\quad 0 \le b \le 1,\quad 0 < c \le 1,\quad -1 < \alpha \le 0,\quad 0 \le \beta\gamma < 1;$$
$$2)\quad -1 \le b \le 0,\quad -1 \le c < 0,\quad 0 \le \alpha < 1,\quad 0 \le \beta\gamma < 1;$$
$$3)\quad 0 \le b \le 1,\quad -1 \le c < 0,\quad -1 < \alpha \le 0,\quad -1 < \beta\gamma \le 0;$$
$$4)\quad -1 \le b \le 0,\quad 0 < c \le 1,\quad 0 \le \alpha < 1,\quad -1 < \beta\gamma \le 0.$$

Under a special choice of the parameters, the spectral radii of the matrices (8) can be calculated in quite a simple way. Thus, for example, if $\beta c = \gamma b$, then

$$\rho(A_0) = \max\{|\lambda| : \lambda^2 - \gamma\lambda - \alpha b = 0\}, \qquad \rho(A_1) = \max\{|\lambda| : \lambda^2 + \gamma\lambda + \alpha b = 0\}.$$

Moreover, the following theorem is valid.

Theorem 1. *Suppose that $\widehat{\rho}$ is the joint spectral radius of the matrices* (8). *Then*

$$\widehat{\rho} = \begin{cases} \sqrt[3]{|\alpha|} & \text{if } b = 0,\ |c| = 1,\ 0 \le |\alpha| < 1, \\ \max\{\sqrt{|\alpha|}, |\gamma|\} & \text{if } |b| = 1,\ 0 \le |\alpha| < 1,\ 0 \le |\gamma| < 1, \\ |\gamma| & \text{if } |a| = 1,\ 0 \le |\gamma| < 1 \end{cases}$$

and $\alpha_\varphi = -\log_2 \widehat{\rho}$.

Proof. By Theorem 3 from [8], the function φ generates a multiresolution analysis in $L^2(\mathbb{R}_+)$, and, by Proposition 1, we have $\alpha_\varphi = -\log_2 \widehat{\rho}$. Suppose that $\beta = c = 1$. Then

$$T_0 e_1 = -T_1 e_1 = \alpha e_3, \qquad T_0 e_2 = -T_1 e_2 = e_1, \qquad T_0 e_3 = T_1 e_3 = e_2.$$

Denote by $\operatorname{mod}(m, 3)$ the remainder in the division of m by 3. By induction, we can verify that

$$T_0^m e_1 = \alpha^{[(m+2)/3]} e_{3-\operatorname{mod}(m+2,3)}, \qquad m \in \mathbb{N},$$

where $[x]$ is the integer part of a number x. Similarly,

$$T_0^m e_2 = \alpha^{[(m+1)/3]} e_{3-\operatorname{mod}(m+4,3)}, \qquad T_0^m e_3 = \alpha^{[m/3]} e_{3-\operatorname{mod}(m+3,3)}, \qquad m \in \mathbb{N}.$$

Hence we find

$$\|T_0^m e_j\| \le C|\alpha|^{m/3}, \qquad j = 1, 2, 3.$$

Moreover, since $T_0 e_j = \pm T_1 e_j$, it follows that similar estimates hold for the vectors $T_{d_1} T_{d_2} \cdots T_{d_m} e_j$. Therefore, for all $m \in \mathbb{N}$,

$$\|T_{d_1} T_{d_2} \cdots T_{d_m}|_{E_1}\| \le C|\alpha|^{m/3}, \qquad d_1, d_2, \dots, d_m \in \{0, 1\},$$

and the estimate $\widehat{\rho} \le \sqrt[3]{|\alpha|}$ holds (see Proposition 2).

Let us prove the reverse inequality. It is well known [8, p. 151] that φ can be expanded in the lacunary Walsh series:

$$\varphi(x) = \frac{1}{4}\chi_{[0,1)}(y)\bigl(1 + a\{w_1(y) + w_3(y) + w_6(y) + \alpha w_{13}(y) + \alpha w_{27}(y) + \alpha w_{54}(y) + \alpha^2 w_{109}(y) + \alpha^2 w_{219}(y) + \alpha^2 w_{438}(y) + \alpha^3 w_{877}(y) + \cdots\}\bigr),$$

where $y = x/4$. Therefore,

$$\varphi(0) = \frac{1}{4}\biggl(1 + \frac{3a}{1-\alpha}\biggr).$$

In addition, if $s = 3k$, then

$$\varphi\left(\frac{1}{2^s}\right) = \frac{1}{4}(1 + a\{3 + 3\alpha + \dots + 3\alpha^{(s-3)/3} - \alpha^{s/3} - \alpha^{(s+3)/3} + \dots\})$$
$$= \frac{1}{4}(1 + a(3 - 4\alpha^{s/3})(1-\alpha)^{-1}).$$

Further, if $s = 3k+1$, then

$$\varphi\left(\frac{1}{2^s}\right) = \frac{1}{4}(1 + a\{3 + 3\alpha + \dots + 3\alpha^{(s-4)/3} + \alpha^{(s-1)/3} - \alpha^{(s+2)/3} - \alpha^{(s+5)/3} + \dots\})$$
$$= \frac{1}{4}(1 + a\{(3(1 - \alpha^{(s-1)/3}) - \alpha^{(s+2)/3})(1-\alpha)^{-1} + \alpha^{(s-1)/3}\})$$

and if $s = 3k+2$, then

$$\varphi\left(\frac{1}{2^s}\right) = \frac{1}{4}(1 + a\{3 + 3\alpha + \dots + 3\alpha^{(s-2)/3} - \alpha^{(s+1)/3} - \alpha^{(s+4)/3} + \dots\})$$
$$= \frac{1}{4}(1 + a(3 - 4\alpha^{(s+1)/3})(1-\alpha)^{-1}).$$

Using these expansions, we obtain

$$\left|\varphi(0) - \varphi\left(\frac{1}{2^s}\right)\right| \le C|\alpha|^{s/3}, \qquad s \in \mathbb{N},$$

and, therefore, $\widehat{\rho} \ge \sqrt[3]{|\alpha|}$. The cases $\beta = -1$, $c = 1$, as well as $\beta = 1$, $c = -1$, and $\beta = c = -1$, are treated in a similar way. Therefore, if $b = 0$, $|c| = 1$, $0 \le |\alpha| < 1$, then $\widehat{\rho} = \sqrt[3]{|\alpha|}$.

Now suppose that $b = 1$. Then

$$T_0e_1 = -T_1e_1 = \alpha e_3, \qquad T_0e_2 = -T_1e_2 = \gamma e_2, \qquad T_0e_3 = T_1e_3 = e_1 + ce_2.$$

Therefore, for m even,

$$T_0^m e_1 = \alpha^{m/2}(e_1 + ce_2) + (\gamma^2\alpha^{m/2-1} + \gamma^4\alpha^{m/2-2} + \dots + \gamma^{m-2}\alpha)ce_2,$$
$$T_0^m e_2 = \gamma^m e_2, \qquad T_0^m e_3 = \alpha^{m/2}e_3 + (\gamma\alpha^{m/2-1} + \gamma^3\alpha^{m/2-2} + \dots + \gamma^{m-1}\alpha)ce_2,$$
$$\|T_0^m e_1\| \le C(|\alpha|^{m/2} + |\gamma|^2|\alpha|^{m/2-1} + |\gamma|^4|\alpha|^{m/2-2} + \dots + |\gamma|^{m-2}|\alpha|)$$
$$\le \begin{cases} Cm|\gamma|^m, & |\alpha| \le |\gamma|^2, \\ Cm|\alpha|^{m/2}, & |\alpha| \ge |\gamma|^2, \end{cases}$$
$$\|T_0^m e_2\| \le C|\gamma|^m, \qquad \|T_0^m e_3\| \le \begin{cases} Cm|\gamma|^m, & |\alpha| \le |\gamma|^2, \\ Cm|\alpha|^{m/2}, & |\alpha| \ge |\gamma|^2. \end{cases}$$

Further, for m odd, we have

$$T_0^m e_1 = \alpha^{(m+1)/2}e_3 + (\gamma\alpha^{(m-1)/2} + \gamma^3\alpha^{(m-3)/2} + \dots + \gamma^{m-2}\alpha)ce_2, \qquad T_0^m e_2 = \gamma^m e_2,$$
$$T_0^m e_3 = \alpha^{(m-1)/2}(e_1 + ce_2) + (\gamma^2\alpha^{(m-3)/2} + \gamma^4\alpha^{(m-5)/2} + \dots + \gamma^{m-1}\alpha)ce_2$$

and, as above,

$$\|T_0^m e_1\| \le C(|\alpha|^{(m+1)/2} + |\gamma|\,|\alpha|^{(m-1)/2} + |\gamma|^3|\alpha|^{(m-3)/2} + \dots + |\gamma|^{m-2}|\alpha|)$$
$$\le \begin{cases} Cm|\gamma|^m, & |\alpha| \le |\gamma|^2, \\ Cm|\alpha|^{m/2}, & |\alpha| \ge |\gamma|^2, \end{cases}$$
$$\|T_0^m e_2\| \le C|\gamma|^m, \qquad \|T_0^m e_3\| \le \begin{cases} Cm|\gamma|^m, & |\alpha| \le |\gamma|^2, \\ Cm|\alpha|^{m/2}, & |\alpha| \ge |\gamma|^2. \end{cases}$$

Similar estimates also hold for the vectors $T_{d_1}T_{d_2}\dots T_{d_m}e_j$. Therefore, the following inequality is valid:

$$\|T_{d_1}T_{d_2}\cdots T_{d_m}|_{E_1}\|^{1/m}\le Cm^{1/m}\max\{\sqrt{|\alpha|},|\gamma|\},\qquad m\in\mathbb{N},$$

and, therefore, $\widehat{\rho}\le\max\{\sqrt{|\alpha|},|\gamma|\}$.

To prove of the reverse inequality, we use formula (5.7) from [8]:

$$\begin{aligned}4\varphi(x)=1+a\big(&w_1(y)+w_2(y)+cw_3(y)+\alpha w_5(y)+\gamma cw_7(y)+\alpha w_{10}(y)+\alpha cw_{11}(y)\\&+\gamma^2cw_{15}(y)+\alpha^2w_{21}(y)+\alpha\gamma cw_{23}(y)+\gamma^3cw_{31}(y)+\alpha^2w_{42}(y)\\&+\alpha^2cw_{43}(y)+\alpha\gamma^2cw_{47}(y)+\gamma^4cw_{63}(y)+\alpha^3w_{85}(y)+\alpha^2\gamma cw_{87}(y)\\&+\alpha\gamma^3cw_{95}(y)+\gamma^5cw_{127}(y)+\alpha^3w_{170}(y)+\alpha^3cw_{171}(y)\\&+\alpha^2\gamma^2cw_{175}(y)+\alpha\gamma^4cw_{191}(y)+\gamma^6cw_{255}(y)+\alpha^4w_{341}(y)+\cdots\big),\end{aligned}\tag{9}$$

where $0\le x<4$ and $y=x/4$. Substituting the value $x=0$ into (9), we obtain

$$\varphi(0)=\frac14\biggl(1+\frac{a}{1-\alpha}\biggl(2+\frac{c}{1-\gamma}\biggr)\biggr).$$

Further, it follows from formula (9) that if s is even, then

$$\varphi\biggl(\frac{1}{2^s}\biggr)=\frac14\biggl(1+\frac{2a}{1-\alpha}\biggl\{1-\alpha^{s/2}-c(\alpha^{s/2}+\gamma^2\alpha^{(s-2)/2}+\cdots+\alpha\gamma^{s-2})-\frac{c(\gamma^s-1/2)}{1-\gamma}\biggr\}\biggr),$$

but if s is odd, then

$$\begin{aligned}\varphi\biggl(\frac{1}{2^s}\biggr)=\frac14\biggl(1+\frac{2a}{1-\alpha}\biggl\{&1-\alpha^{(s+1)/2}-c\gamma(\alpha^{(s-1)/2}+\gamma^2\alpha^{(s-3)/2}+\cdots+\alpha\gamma^{s-3})\\&-\frac{c(\gamma^s-1/2)}{1-\gamma}\biggr\}\biggr).\end{aligned}$$

Therefore,

$$\biggl|\varphi(0)-\varphi\biggl(\frac{1}{2^s}\biggr)\biggr|\le Cs(\max\{\sqrt{|\alpha|},|\gamma|\})^s,\qquad s\in\mathbb{N},$$

and hence $\widehat{\rho}\ge\max\{\sqrt{|\alpha|},|\gamma|\}$. Thus, $\widehat{\rho}=\max\{\sqrt{|\alpha|},|\gamma|\}$. The case $b=-1$ is treated in a similar way. Necessary estimates for the case $|a|=1$, $0\le|\gamma|<1$ are obtained in Example 4.4 from [11]. The theorem is proved. □

Recall that a function $f\colon\mathbb{R}_+\to\mathbb{R}$ is said to be *binary entire of order* n if it is constant on binary intervals of rank n. Any compactly supported scaling function in $L^2(\mathbb{R}_+)$ is either binary entire or of finite smoothness and can be effectively estimated from above (see [8, Sec. 7]). Combining this with Theorem 1, we see that, in the case $\alpha=\gamma=0$, the solution of Eq. (2) is a binary entire function (see also Example 3 from [8]).

3. ESTIMATES OF α_φ FOR $n=4$

Suppose that the coefficients of the scaling equation (2) are defined using formulas (3) for

$$b_0=1,\qquad b_l=d_l,\qquad b_8=0,\qquad b_{l+8}=\gamma_l,\quad 1\le l\le 7,\tag{10}$$

where $|d_l|^2+|\gamma_l|^2=1$. Then the compactly supported L^2-solution φ of Eq. (2) can be expanded in the lacunary Walsh series

$$\varphi(x)=\frac18\chi_{[0,1)}\biggl(\frac{x}{8}\biggr)\biggl(1+\sum_{l\in\mathbb{N}(4)}a_lw_l\biggl(\frac{x}{8}\biggr)\biggr),\qquad x\in\mathbb{R}_+,\tag{11}$$

(see [8, Sec. 5], [11, Remark 1.2]). In this expansion, $\mathbb{N}(4) = \{1, 2, \dots, 7\} \cup \mathbb{N}_0(4)$, where $\mathbb{N}_0(4)$ is the set of all natural numbers $l > 7$ whose binary expansion

$$l = \sum_{j=0}^{k} \mu_j 2^j, \qquad \mu_j \in \{0, 1\}, \quad \mu_k \neq 0 \quad k = k(l) \in \mathbb{Z}_+, \tag{12}$$

has no collection $(0, 0, 0, 1)$ among the ordered collections $(\mu_j, \mu_{j+1}, \mu_{j+2}, \mu_{j+3})$, while the coefficients a_l are defined as follows. Suppose that

$$\beta(i_1, i_2, i_3, i_4) = b_s \qquad \text{if} \quad s = i_1 2^0 + i_2 2^1 + i_2 2^2 + i_3 2^3, \quad i_j \in \{0, 1\},$$

where the b_s are parameters given by formulas (10). Let us express each integer $l \in \mathbb{N}(4)$ in the form (12), and let

$$\begin{aligned}
&a_l = \beta(\mu_0, 0, 0, 0) && \text{if} \quad k(l) = 0;\\
&a_l = \beta(\mu_1, 0, 0, 0)\beta(\mu_0, \mu_1, 0, 0, 0) && \text{if} \quad k(l) = 1;\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&a_l = \beta(\mu_k, 0, 0, 0)\beta(\mu_{k-1}, \mu_k, 0, 0, 0) \cdots \beta(\mu_0, \mu_1, \mu_2, \mu_3) && \text{if} \quad k = k(l) \geq 3.
\end{aligned}$$

Note that, in the last product, the indices of each multiplier, beginning with the second, are obtained by "shifting" the indices of the preceding multiplier by one position to the right and by adding, to the liberated first place, one new digit from the binary expansion (12) of the number l.

In view of our choice of the parameters b_s, the mask

$$m(\omega) = \frac{1}{2} \sum_{k=0}^{15} c_k w_k(\omega) \tag{13}$$

satisfies the conditions

$$m(0) = 1, \qquad |m(\omega)|^2 + |m(\omega + 1/2)|^2 = 1 \quad \text{for all } \omega \in \mathbb{R}_+. \tag{14}$$

Suppose that the set M is the union of some of the intervals $[s/8, (s + 1)/8)$, $s = 1, 2, \dots, 7$, or coincides with one of them. If the mask (13) vanishes at each point of the set

$$\left(\left\{\frac{\omega}{2} : \omega \in M\right\} \cup \left\{\frac{\omega + 1}{2} : \omega \in M\right\}\right) \setminus M,$$

then the set M is said to be *blocking*. Taking (14) into account and applying Theorem 3 from [8], we see that the solution φ of Eq. (2) generates a multiresolution analysis in $L^2(\mathbb{R}_+)$ if and only if the mask (13) has no blocking sets. Let us present a complete list of collections of values of the parameters for which the mask (13) has blocking sets and, for each of these collections, let us indicate one blocking set:

1) $d_1 = 0$, $M = [1/8, 1)$;

2) $d_7 = 0$, $M = [7/8, 1)$;

3) $d_2 = d_3 = 0$, $M = [1/4, 1/2) \cup [5/8, 1)$;

4) $d_2 = \gamma_5 = 0$, $M = [1/4, 3/8) \cup [5/8, 3/4)$;

5) $d_3 = d_5 = 0$, $M = [3/8, 1/2) \cup [5/8, 1)$.

In addition, note that, for $n = 4$, the vectors

$$\begin{aligned}
&e_1 = (1, -1, 1, -1, 1, -1, 1, -1)^t, && e_2 = (1, 1, -1, -1, -1, -1, 1, 1)^t,\\
&e_3 = (1, 1, 1, 1, -1, -1, -1, -1)^t, && e_4 = (1, -1, 1, -1, -1, 1, -1, 1)^t,\\
&e_5 = (1, 1, -1, -1, 1, 1, -1, -1)^t, && e_6 = (1, -1, -1, 1, 1, -1, -1, 1)^t,\\
&e_7 = (1, -1, -1, 1, 1, -1, 1, -1)^t
\end{aligned} \tag{15}$$

constitute a basis in the space E_1.

In what follows, T_0 and T_1 are matrices defined by formulas (3) and (5) under condition (10). We introduce the matrices

$$B_0 = \begin{pmatrix} 0 & -d_7 & 0 & 0 & d_4 & \gamma_4 & 0 \\ 0 & 0 & d_3 & \gamma_3 & 0 & 0 & 0 \\ \gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_7 & 0 & 0 & d_5 & \gamma_5 & 0 \\ 0 & 0 & d_2 & \gamma_2 & 0 & 0 & 0 \\ 0 & d_6 - d_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2d_7 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 & -d_7 & 0 & 0 & d_4 & -\gamma_4 & 0 \\ 0 & 0 & d_3 & -\gamma_3 & 0 & 0 & 0 \\ \gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_7 & 0 & 0 & d_5 & -\gamma_5 & 0 \\ 0 & 0 & d_2 & -\gamma_2 & 0 & 0 & 0 \\ 0 & d_6 - d_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2d_7 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and denote by Γ the matrix of dimensions 7×7 whose first six columns consist of zeros and the last column coincides with the vector

$$(\gamma_4 - \gamma_7, -\gamma_3, \gamma_1, \gamma_5 + \gamma_7, -\gamma_2, \gamma_6 - \gamma_7, \gamma_7)^t.$$

Then the linear transforms

$$y = T_0 x, \quad y = T_1 x, \qquad x \in E_1,$$

have the matrices

$$A_0 = B_0 + \frac{1}{2}\Gamma, \qquad A_1 = B_1 - \frac{1}{2}\Gamma$$

respectively, in the basis (15). Under the assumptions of Proposition 1, the formula $\widehat{\rho}(A_0, A_1) = 2^{-\alpha_\varphi}$ is valid. Let us point out three cases in which the quantity α_φ can be calculated exactly.

Theorem 2. *Suppose that the function φ is given by the expansion* (11). *Then*

1) $\alpha_\varphi = \dfrac{1}{2}\log_2 \dfrac{1}{|\gamma_2|}$ *if* $\gamma_2 \neq 0$ *and* $\gamma_1 = \gamma_5 = \gamma_7 = 0$;

2) $\alpha_\varphi = \dfrac{1}{2}\log_2 \dfrac{1}{|d_5\gamma_2|}$ *if* $d_5\gamma_2 \neq 0$ *and* $\gamma_1 = \gamma_3 = \gamma_7 = 0$;

3) $\alpha_\varphi = \log_2 \dfrac{1}{|\gamma_7|}$ *if* $\gamma_7 \neq 0$

and one of the following conditions holds:

a) $\gamma_1 = \gamma_2 = \gamma_3 = 0$, b) $\gamma_1 = \gamma_2 = \gamma_5 = 0$, c) $d_5 = \gamma_1 = \gamma_3 = 0$.

Proof. 1^0. Suppose that $d_1 = d_5 = d_7 = 1$ and $\gamma_2 \neq 0$. Then the mask (13) has no blocking sets and, therefore, the function φ generates a multiresolution analysis in $L^2(\mathbb{R}_+)$. By (11), we have

$$\begin{aligned} 8\varphi(x) = \chi_{[0,1)}(y)\bigl(1 &+ w_1(y) + d_2w_2(y) + d_3w_3(y) + d_2d_4w_4(y) + d_2w_5(y) + d_3d_6w_6(y) \\ &+ d_3w_7(y) + d_2\gamma_2w_{10}(y) + d_2\gamma_3w_{11}(y) + d_3d_6\gamma_4w_{12}(y) \\ &+ d_3\gamma_6w_{14}(y) + d_2d_4\gamma_2w_{20}(y) + d_2\gamma_2w_{21}(y) + d_2d_6\gamma_3w_{22}(y) \\ &+ d_2\gamma_3w_{23}(y) + d_3\gamma_4\gamma_6w_{28}(y) + d_2\gamma_2^2w_{42}(y) + d_2\gamma_2\gamma_3w_{43}(y) \\ &+ d_2d_6\gamma_3\gamma_4w_{44}(y) + d_2\gamma_3\gamma_6w_{46}(y) + d_2d_4\gamma_2^2w_{84}(y) + \cdots\bigr), \end{aligned}$$

where $y = x/8$. Let

$$G_1 := 2 + 2d_2 + 2d_3 + d_2d_4 + d_3d_6 + 2d_2\gamma_2 + 2d_2\gamma_3 + d_3d_6\gamma_4 + d_3\gamma_6 + d_2d_4\gamma_3 + d_2d_6\gamma_3 + d_3\gamma_4\gamma_6.$$

Then

$$8\varphi(0) = G_1 + \frac{d_2}{1-\gamma_2}(2\gamma_2^2 + 2\gamma_2\gamma_3 + d_6\gamma_3\gamma_4 + \gamma_3\gamma_6 + d_4\gamma_2^2 + d_6\gamma_2\gamma_3 + \gamma_3\gamma_4\gamma_6).$$

In addition, for any odd number $s \geq 5$,

$$8\varphi\left(\frac{1}{2^s}\right) = G_1 + \frac{d_2}{1-\gamma_2}\big((\gamma_2^2 + \gamma_2\gamma_3 + d_6\gamma_3\gamma_4 + \gamma_3\gamma_6)(1 - 2\gamma_2^{(s-3)/2}) + (\gamma_2^2 + d_4\gamma_2^2 + d_6\gamma_2\gamma_3 + \gamma_2\gamma_3 + \gamma_3\gamma_4\gamma_6)\big)$$

and, for any even number $s \geq 6$,

$$8\varphi\left(\frac{1}{2^s}\right) = G_1 + \frac{d_2}{1-\gamma_2}\big(\gamma_2^2 + \gamma_2\gamma_3 + d_6\gamma_3\gamma_4 + \gamma_3\gamma_6 + (\gamma_2^2 + d_4\gamma_2^2 + d_6\gamma_2\gamma_3 + \gamma_2\gamma_3 + \gamma_3\gamma_4\gamma_6)(1 - 2\gamma_2^{(s-4)/2})\big).$$

Therefore,

$$\left|\varphi(0) - \varphi\left(\frac{1}{2^s}\right)\right| \leq C|\gamma_2|^{s/2},$$

and hence

$$\alpha_\varphi \leq \frac{1}{2}\log_2\frac{1}{|\gamma_2|}\,.$$

To prove the reverse inequality, note that, for the vectors (15) and for all $i, j, k, l \in \{0, 1\}$,

$$T_ie_1 = T_iT_jT_kT_le_2 = 0, \qquad T_ie_4 = \pm(\gamma_3e_2 + \gamma_2e_5), \qquad T_iT_je_5 = \pm T_ie_4.$$

Therefore,

$$\|T_{d_1}T_{d_2}\cdots T_{d_m}e_i\| \leq C|\gamma_2|^{m/2} \qquad \text{for} \quad i = 1, 2, 4, 5.$$

Further, for all $i, j, k, l \in \{0, 1\}$, the following relations hold:

$$T_iT_je_6 = 0, \qquad T_ie_3 = d_3e_2 + d_2e_5,$$

$$T_iT_jT_kT_le_4 = \pm\gamma_2^2(d_4e_1 + e_4) + \gamma_3(T_iT_jT_ke_2 \pm \gamma_2T_ie_2), \qquad T_iT_jT_ke_7 = \pm\frac{1}{2}T_iT_je_4.$$

Using these equalities for an arbitrary vector $w \in E_1$ and any $d_1, d_2, \dots, d_m \in \{0, 1\}$, we obtain

$$\|T_{d_1}T_{d_2}\cdots T_{d_m}w\| \leq C|\gamma_7|^m\|w\|.$$

Therefore, for any $m \in \mathbb{N}$, the following estimate holds:

$$\|T_{d_1}T_{d_2}\cdots T_{d_m}|_{E_1}\| \leq C|\gamma_2|^{m/2}$$

and, by Proposition 2,

$$\alpha_\varphi \geq \frac{1}{2}\log_2\frac{1}{|\gamma_2|}\,.$$

Thus, if $d_1 = d_5 = d_7 = 1$ and $\gamma_2 \neq 0$, then

$$\alpha_\varphi = \frac{1}{2}\log_2\frac{1}{|\gamma_2|}\,.$$

2^0. Suppose that $d_1 = d_3 = d_7 = 1$ and $d_5\gamma_2 \neq 0$. Then the mask (13) has no blocking sets and the function φ generates a multiresolution analysis in $L^2(\mathbb{R}_+)$. Moreover, by (11), we have

$$8\varphi(x) = \chi_{[0,1)}(y)\big(1 + w_1(y) + d_2w_2(y) + w_3(y) + d_2d_4w_4(y) + d_2d_5w_5(y) + d_6w_6(y) + w_7(y) + d_2d_5\gamma_2w_{10}(y) + d_6\gamma_4w_{12}(y) + d_6\gamma_5w_{13}(y) + \gamma_6w_{14}(y)$$

$$+ d_2d_4d_5\gamma_2w_{20}(y) + d_2d_5^2\gamma_2w_{21}(y) + d_6\gamma_2\gamma_5w_{26}(y) + \gamma_4\gamma_6w_{28}(y)$$
$$+ \gamma_5\gamma_6w_{29}(y) + d_2d_5^2\gamma_2^2w_{42}(y) + d_4d_6\gamma_2\gamma_5w_{52}(y)$$
$$+ d_5d_6\gamma_2\gamma_5w_{53}(y) + \gamma_2\gamma_5\gamma_6w_{58}(y) + d_2d_4d_5^2\gamma_2^2w_{84}(y) + \cdots\Big),$$

where $y = x/8$. Let

$$G_2 := 4 + d_2 + d_2d_4 + d_2d_5 + d_6 + d_2d_5\gamma_2 + d_6\gamma_4 + d_6\gamma_5 + \gamma_6 + d_2d_4d_5\gamma_2 + d_2d_5^2\gamma_2 + d_6\gamma_2\gamma_5 + \gamma_4\gamma_6 + \gamma_5\gamma_6.$$

Then

$$8\varphi(0) = G_2 + \frac{\gamma_2}{1-d_5\gamma_2}(d_2d_5^2\gamma_2 + d_2\gamma_4d_5^2\gamma_2 + d_2d_5^3\gamma_2 + d_4d_6\gamma_5 + d_5d_6\gamma_5 + \gamma_5\gamma_6 + d_5d_6\gamma_2\gamma_5 + d_4\gamma_5\gamma_6 + d_5\gamma_5\gamma_6).$$

For any odd number $s \ge 5$,

$$8\varphi\left(\frac{1}{2^s}\right) = G_2 + \frac{\gamma_2}{1-d_5\gamma_2}(d_2d_5^2\gamma_2 + \gamma_5\gamma_6 + d_5d_6\gamma_2\gamma_5) \times (1 - 2(d_5\gamma_2)^{(s-3)/2} + 2(d_5\gamma_2)^{(s-3)/2}) + (d_4d_6\gamma_5 + d_5d_6\gamma_5 + d_4\gamma_5\gamma_6 + d_5\gamma_5\gamma_6) \times (1 - 2(d_5\gamma_2)^{(s-1)/2} + d_2d_4d_5^2\gamma_2 + d_2d_5^3\gamma_2).$$

For any even $s \ge 6$,

$$8\varphi\left(\frac{1}{2^s}\right) = G_2 + \frac{\gamma_2}{1-d_5\gamma_2}\Big(d_2d_5^2\gamma_2 + (d_4d_6\gamma_5 + d_5d_6\gamma_5)(1 - 2(d_5\gamma_2)^{(s-2)/2}) + (d_2d_4d_5^2\gamma_2 + d_2d_5^3\gamma_2 + d_4\gamma_5\gamma_6 + d_5\gamma_5\gamma_6) \times (1 - 2(d_5\gamma_2)^{(s-4)/2}) + \gamma_5\gamma_6(1 - 2(d_5\gamma_2)^{(s-2)/2} + 2(d_5\gamma_2)^{s/2}) + d_5d_6\gamma_2\gamma_5(1 - 2(d_5\gamma_2)^{(s-4)/2}) + 2(d_5\gamma_2)^{(s-2)/2}\Big).$$

Therefore,

$$\left|\varphi(0) - \varphi\left(\frac{1}{2^s}\right)\right| \le C|d_5\gamma_2|^{s/2} \qquad \text{and} \qquad \alpha_\varphi \le \frac{1}{2}\log_2\frac{1}{|d_5\gamma_2|}.$$

Now note that, for the vectors (15) and any $i, j \in \{0,1\}$, we have

$$T_ie_1 = 0, \qquad T_iT_je_5 = \pm\gamma_2d_5e_5, \qquad T_ie_4 = \pm\gamma_2e_5$$

and, for $i = 1, 4, 5$, the following estimate is valid:

$$\|T_{d_1}T_{d_2}\cdots T_{d_m}e_i\| \le C|d_5\gamma_2|^{m/2}.$$

In addition, for all $i, j, k, l \in \{0,1\}$, the following relations hold:

$$T_iT_jT_kT_le_2 = \pm\gamma_2d_5T_ie_4 \pm (d_6-1)\gamma_2\gamma_5T_ie_5 + T_iT_jT_ke_7, \qquad T_ie_3 = e_2 + d_2e_5,$$
$$T_iT_je_6 = \pm\gamma_5T_ie_4, \qquad T_iT_jT_ke_7 = \pm\frac{1}{2}(\pm\gamma_2\gamma_5T_ie_5 + (\pm\gamma_5\gamma_6 - d_5)T_ie_4).$$

Using these equalities, for an arbitrary vector $w \in E_1$ and any $d_1, d_2, \dots, d_m \in \{0,1\}$, we obtain

$$\|T_{d_1}T_{d_2}\cdots T_{d_m}w\| \le C|\gamma_7|^m\|w\|.$$

Hence, for any $m \in \mathbb{N}$, the following estimate holds:

$$\|T_{d_1}T_{d_2}\cdots T_{d_m}|_{E_1}\| \le C|d_5\gamma_2|^{m/2}$$

and, by Proposition 2,

$$\alpha_\varphi \geq \frac{1}{2}\log_2\frac{1}{|d_5\gamma_2|}\,.$$

Thus, if $d_1 = d_3 = d_7 = 1$ and $d_5\gamma_2 \neq 0$, then

$$\alpha_\varphi = \frac{1}{2}\log_2\frac{1}{|d_5\gamma_2|}\,.$$

3^0. Suppose that $d_1 = d_2 = d_3 = 1$ and $\gamma_7 \neq 0$. Then the mask (13) has no blocking sets and the function φ generates a multiresolution analysis in $L^2(\mathbb{R}_+)$. Applying formula (11), we obtain

$$\begin{aligned}8\varphi(x) = \chi_{[0,1)}(y)\big(1 &+ w_1(y) + w_2(y) + w_3(y) + d_4w_4(y) + d_5w_5(y) + d_6w_6(y) + d_7w_7(y)\\ &+ d_6\gamma_4w_{12}(y) + d_6\gamma_5w_{13}(y) + d_7\gamma_6w_{14}(y) + d_7\gamma_7w_{15}(y)\\ &+ d_7\gamma_4\gamma_6w_{28}(y) + d_7\gamma_5\gamma_6w_{29}(y) + d_7\gamma_6\gamma_7w_{30}(y) + d_7\gamma_7^2w_{31}(y)\\ &+ d_7\gamma_4\gamma_6\gamma_7w_{60}(y) + d_7\gamma_5\gamma_6\gamma_7w_{61}(y) + d_7\gamma_6\gamma_7^2w_{62}(y)\\ &+ d_7\gamma_7^3w_{63}(y) + d_7\gamma_4\gamma_6\gamma_7^2w_{124}(y) + d_7\gamma_5\gamma_6\gamma_7^2w_{125}(y) + \cdots\big),\end{aligned}$$

where $y = x/8$. Let

$$G_3 := 4 + d_4 + d_5 + d_6 + d_6\gamma_4 + d_6\gamma_5.$$

Then

$$8\varphi(0) = G_3 + \frac{d_7}{1-\gamma_7}(1 + \gamma_6(1 + \gamma_4 + \gamma_5)).$$

In addition, for any integer $s \geq 2$,

$$8\varphi\left(\frac{1}{2^s}\right) = G_3 + \frac{d_7}{1-\gamma_7}\big(1 - 2\gamma_7^s + (1 - 2\gamma_7^{s-2})(d_6\gamma_5 + \gamma_4\gamma_6) + \gamma_6(1 - 2\gamma_7^{s-1})\big).$$

Therefore,

$$\left|\varphi(0) - \varphi\left(\frac{1}{2^s}\right)\right| \leq C|\gamma_7|^s.$$

For all $i, j, k \in \{0, 1\}$, the following relations hold:

$$T_iT_jT_ke_1 = T_iT_jT_ke_4 = T_iT_jT_ke_5 = T_iT_jT_ke_6 = 0,$$
$$T_iT_jT_ke_2 = \pm d_7\gamma_7T_iT_je_7, \qquad T_iT_jT_ke_3 = (d_6 - d_7)(\gamma_4e_1 + \gamma_5e_4) + 2d_7T_ie_7,$$
$$T_iT_jT_ke_7 = \pm\gamma_7T_iT_je_7, \qquad T_iT_je_7 = \pm\frac{1}{2}(\pm(\gamma_6 - \gamma_7)(\gamma_4e_1 + \gamma_5e_4) + 2\gamma_7T_ie_7).$$

Hence, for an arbitrary vector $w \in E_1$ and any $d_1, d_2, \ldots, d_m \in \{0, 1\}$,

$$\|T_{d_1}T_{d_2}\cdots T_{d_m}w\| \leq C|\gamma_7|^m\|w\|.$$

Therefore, for any $m \in \mathbb{N}$, the following estimate holds:

$$\|T_{d_1}T_{d_2}\cdots T_{d_m}|_{E_1}\| \leq C|\gamma_7|^m$$

and, by Proposition 2, we have

$$\alpha_\varphi \geq \log_2\frac{1}{|\gamma_7|}\,.$$

Thus, if $d_1 = d_2 = d_3 = 1$ and $\gamma_7 \neq 0$, then

$$\alpha_\varphi = \log_2\frac{1}{|\gamma_7|}\,.$$

4^0. Suppose that $d_1 = d_2 = d_5 = 1$ and $\gamma_7 \neq 0$. Then the mask (13) has no blocking sets and the function φ generates a multiresolution analysis in $L^2(\mathbb{R}_+)$. In that case,

$$\begin{aligned}8\varphi(x) = \chi_{[0,1)}(y)\bigl(1 &+ w_1(y) + w_2(y) + d_3w_3(y) + d_4w_4(y) + w_5(y) + d_3d_6w_6(y)\\ &+ d_3d_7w_7(y) + \gamma_3w_{11}(y) + d_3d_6\gamma_4w_{12}(y) + d_3d_7\gamma_6w_{14}(y)\\ &+ d_3d_7\gamma_7w_{15}(y) + d_6\gamma_3w_{22}(y) + d_7\gamma_3w_{23}(y) + d_3d_7\gamma_4\gamma_6w_{28}(y)\\ &+ d_3d_7\gamma_6\gamma_7w_{30}(y) + d_3d_7\gamma_7^2w_{31}(y) + d_6\gamma_3\gamma_4w_{44}(y)\\ &+ d_7\gamma_3\gamma_6w_{46}(y) + d_7\gamma_3\gamma_7w_{47}(y) + d_3d_7\gamma_4\gamma_6\gamma_7w_{60}(y)\\ &+ d_3d_7\gamma_6\gamma_7^2w_{62}(y) + \cdots\bigr),\end{aligned}$$

where $y = x/8$.

Suppose that

$$\begin{aligned}G_4 := 4 &+ d_3 + d_4 + d_3d_6 + d_3d_7 + \gamma_3 + d_3d_6\gamma_4 + d_3d_7\gamma_6 + d_3d_7\gamma_7\\ &+ d_6\gamma_3 + d_7\gamma_3 + d_3d_7\gamma_4\gamma_6 + d_3d_7\gamma_6\gamma_7 + d_3d_7\gamma_7^2 + d_6\gamma_3\gamma_4\\ &+ d_7\gamma_3\gamma_6 + d_7\gamma_3\gamma_7 + d_3d_7\gamma_4\gamma_6\gamma_7 + d_3d_7\gamma_6\gamma_7^2.\end{aligned}$$

Then

$$8\varphi(0) = G_4 + \frac{d_7}{1-\gamma_7}(d_3\gamma_7^3 + \gamma_3\gamma_4\gamma_6 + \gamma_3\gamma_6\gamma_7 + \gamma_3\gamma_7^2 + d_3\gamma_4\gamma_6\gamma_7^2 + d_3\gamma_6\gamma_7^3).$$

For any integer $s \ge 5$,

$$\begin{aligned}8\varphi\biggl(\frac{1}{2^s}\biggr) = G_4 + \frac{d_7}{1-\gamma_7}\bigl(&d_3\gamma_7^3(1-2\gamma_7^{s-3})\\ &+ \gamma_3(\gamma_4\gamma_6 + \gamma_6\gamma_7 + \gamma_7^2)(1 - 2\gamma_7^{s-4} + 2\gamma_7^{s-3} - 2\gamma_7^{s-2})\\ &+ d_3\gamma_6\gamma_7^2(\gamma_4 + \gamma_7)(1 - 2\gamma_7^{s-4})\bigr).\end{aligned}$$

It is readily seen that

$$\left|\varphi(0) - \varphi\biggl(\frac{1}{2^s}\biggr)\right| \le C|\gamma_7|^s,$$

and hence

$$\alpha_\varphi \le \log_2 \frac{1}{|\gamma_7|}\,.$$

Further, just as above, for all $i, j, k \in \{0, 1\}$,

$$\begin{gathered}T_ie_1 = T_iT_je_6 = 0, \qquad T_iT_jT_ke_2 = \pm d_7\gamma_3T_ie_2 + 2d_7T_iT_je_7,\\ T_iT_jT_ke_3 = d_3T_iT_je_2 \pm \gamma_3e_2, \qquad T_ie_4 = T_iT_je_5 = \pm\gamma_3e_2,\\ T_iT_je_7 = \pm\frac{1}{2}(\gamma_3T_ie_2 + \gamma_7T_ie_4 \pm (\gamma_6 - \gamma_7)\gamma_4e_1 + \gamma_7T_ie_7).\end{gathered}$$

Hence, for an arbitrary vector $w \in E_1$ and any $d_1, d_2, \dots, d_m \in \{0, 1\}$, we obtain

$$\|T_{d_1}T_{d_2}\cdots T_{d_m}w\| \le C|\gamma_7|^m\|w\|.$$

Therefore, for any $m \in \mathbb{N}$, the following estimate holds:

$$\|T_{d_1}T_{d_2}\cdots T_{d_m}|_{E_1}\| \le C|\gamma_7|^m$$

and, by Proposition 2, we have

$$\alpha_\varphi \ge \log_2 \frac{1}{|\gamma_7|}\,.$$

Thus, if $d_1 = d_2 = d_5 = 1$ and $\gamma_7 \neq 0$, then

$$\alpha_\varphi = \log_2 \frac{1}{|\gamma_7|}.$$

5^0. Suppose that $d_1 = d_3 = \gamma_5 = 1$ and $\gamma_7 \neq 0$. Then the mask (13) has no blocking sets and the function φ generates a multiresolution analysis in $L^2(\mathbb{R}_+)$. In addition,

$$\begin{aligned}8\varphi(x) = \chi_{[0,1)}(y)\big(1 &+ w_1(y) + d_2w_2(y) + w_3(y) + d_2d_4w_4(y) + d_6w_6(y) + d_7w_7(y)\\
&+ d_6\gamma_4w_{12}(y) + d_6w_{13}(y) + d_7\gamma_6w_{14}(y) + d_7\gamma_7w_{15}(y)\\
&+ d_6\gamma_2w_{26}(y) + d_7\gamma_4\gamma_6w_{28}(y) + d_7\gamma_6w_{29}(y) + d_7\gamma_6\gamma_7w_{30}(y)\\
&+ d_7\gamma_7^2w_{31}(y) + d_4d_6\gamma_2w_{52}(y) + d_7\gamma_2\gamma_6w_{58}(y) + d_7\gamma_4\gamma_6\gamma_7w_{60}(y)\\
&+ d_7\gamma_6\gamma_7w_{61}(y) + d_7\gamma_6\gamma_7^2w_{62}(y) + \cdots\big),\end{aligned}$$

where $y = x/8$. Let

$$\begin{aligned}G_5 := 3 + d_2 + d_2d_4 + 2d_6 + d_7 + d_6\gamma_4 + 2d_7\gamma_6 + d_7\gamma_7 + d_6\gamma_2 + d_7\gamma_4\gamma_6\\
+ d_7\gamma_6\gamma_7 + 2d_7\gamma_7^2 + d_4d_6\gamma_2 + d_7\gamma_2\gamma_6 + d_7\gamma_4\gamma_6\gamma_7 + d_7\gamma_6\gamma_7^2.\end{aligned}$$

Then

$$8\varphi(0) = G_5 + \frac{d_7}{1-\gamma_7}(\gamma_7^3 + d_4\gamma_2\gamma_6 + \gamma_2\gamma_6\gamma_7 + \gamma_4\gamma_6\gamma_7^2 + \gamma_6\gamma_7^2 + \gamma_6\gamma_7^3).$$

For any integer $s \geq 5$, we obtain

$$8\varphi\left(\frac{1}{2^s}\right) = G_5 + \frac{d_7}{1-\gamma_7}\big(\gamma_7^3(1-\gamma_7^{s-3}) + \gamma_6(d_4\gamma_2 + \gamma_2\gamma_7 + \gamma_4\gamma_7^2 + \gamma_7^2 + \gamma_7^3)(1-2\gamma_7^{s-4})\big),$$

and the following estimate is valid:

$$\left|\varphi(0) - \varphi\left(\frac{1}{2^s}\right)\right| \leq C|\gamma_7|^s.$$

Hence

$$\alpha_\varphi \leq \log_2 \frac{1}{|\gamma_7|}.$$

Further,

$$\begin{gathered}T_ie_1 = T_iT_jT_ke_4 = T_iT_je_5 = T_iT_jT_kT_le_6 = 0,\\
T_ie_2 = -d_7e_1 + d_7e_4 + (d_6 - d_7)e_6 + 2d_7, \qquad T_ie_3 = e_2 + d_2e_6,\\
T_ie_7 = \pm\frac{1}{2}((\gamma_4 - \gamma_7)e_1 - (\gamma_7 + 1)e_4 - \gamma_2e_5 + (\gamma_6 - \gamma_7)e_6 + \gamma_7e_7).\end{gathered}$$

Therefore, for an arbitrary vector $w \in E_1$ and any $d_1, d_2, \dots, d_m \in \{0, 1\}$, the following estimate is valid:

$$\|T_{d_1}T_{d_2}\cdots T_{d_m}w\| \leq C|\gamma_7|^m\|w\|$$

and, for any $m \in \mathbb{N}$, the following inequality holds:

$$\|T_{d_1}T_{d_2}\cdots T_{d_m}|_{E_1}\| \leq C|\gamma_7|^m.$$

Hence, using Proposition 2, we obtain the estimate

$$\alpha_\varphi \geq \log_2 \frac{1}{|\gamma_7|}.$$

Thus, if $\gamma_5 = d_1 = d_3 = 1$ and $\gamma_7 \neq 0$, then

$$\alpha_\varphi = \log_2 \frac{1}{|\gamma_7|}.$$

The theorem is proved. □

In conclusion, let us point out three cases in which $\widehat{\rho}(A_0, A_1) = 0$ and the solution of Eq. (2) is a binary entire function. Suppose that $y = x/8$ and $x \in [0, 8)$. Then, setting

$$W(y) := w_1(y) + d_2w_2(y) + d_3w_3(y) + d_2d_4w_4(y) + d_3d_6w_6(y) + d_3d_7w_7(y),$$

we obtain the following three expansions from formula (11):

1) If $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_7 = 0$, then

$$\begin{aligned}8\varphi(x) = 1 + d_1\{W(y) &+ d_2d_5w_5(y) + d_3d_6\gamma_4w_{12}(y) + d_3d_6\gamma_5w_{13}(y)\\ &+ d_3d_7\gamma_6w_{14}(y) + d_3d_7\gamma_4\gamma_6w_{28}(y) + d_3d_7\gamma_5\gamma_6w_{29}(y)\}.\end{aligned}$$

2) If $\gamma_1 = \gamma_2 = \gamma_5 = \gamma_7 = 0$, then

$$\begin{aligned}8\varphi(x) = 1 + d_1\{W(y) &+ d_2d_5w_5(y) + d_2d_5\gamma_3w_{11}(y) + d_3d_6\gamma_4w_{12}(y)\\ &+ d_3d_7\gamma_6w_{14}(y) + d_2d_5d_6\gamma_3w_{22}(y) + d_2d_5d_7\gamma_3w_{23}(y)\\ &+ d_3d_7\gamma_4\gamma_6w_{28}(y) + d_2d_5d_6\gamma_3\gamma_4w_{44}(y)\\ &+ d_2d_5d_7\gamma_3\gamma_6w_{46}(y) + d_2d_5d_7\gamma_3\gamma_4\gamma_6w_{92}(y)\}.\end{aligned}$$

3) If $d_5 = \gamma_1 = \gamma_3 = \gamma_7 = 0$, then

$$\begin{aligned}8\varphi(x) = 1 + d_1\{W(y) &+ d_3d_6\gamma_4w_{12}(y) + d_3d_7\gamma_6w_{14}(y) + d_3d_6\gamma_2\gamma_5w_{26}(y)\\ &+ d_3d_7\gamma_4\gamma_6w_{28}(y) + d_3d_7\gamma_5\gamma_6w_{29}(y) + d_3d_4d_6\gamma_2\gamma_5w_{52}(y)\\ &+ d_3d_7\gamma_2\gamma_5\gamma_6w_{58}(y) + d_3d_4d_7\gamma_2\gamma_5\gamma_6w_{116}(y)\}.\end{aligned}$$

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