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Biorthogonal Wavelets on Vilenkin Groups

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Abstract—We describe algorithms for constructing biorthogonal wavelet systems and refinable functions whose masks are generalized Walsh polynomials. We give new examples of biorthogonal sets of compactly supported wavelets on Vilenkin groups.

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It is well-known that orthogonal compactly supported wavelets on Vilenkin groups can be represented as lacunary series in generalized Walsh functions. In this paper we propose algorithms for constructing biorthogonal wavelet systems and refinable functions whose masks are generalized Walsh polynomials. We also give several new examples of biorthogonal wavelet systems.

1. PRELIMINARIES

The foundations of the theory of Walsh series and transforms as a branch of modern harmonic analysis are expounded in the monographs [1–3]. While the harmonics e^{ikt} are characters of the group of rotations of the circle, the Walsh functions are characters of the Cantor dyadic group. Orthogonal wavelets and the corresponding refinable functions representable as lacunary Walsh series were studied in [4–10]. For $p \geq 2$, the Vilenkin group G can be defined as the weak direct product of a countable set of cyclic groups of order p (in the case of $p = 2$, G is isomorphic to the Cantor group). A peculiarity of the construction of wavelets on the Cantor and Vilenkin groups is associated with the fact that these groups (as well as the additive group of the p-adic number field) contain open compact subgroups (see [11]). In the present paper, by analogy with compactly supported biorthogonal wavelets on $\mathbb{R}$, which are determined by appropriately chosen trigonometric polynomials (see, e.g., [12, Section 8.3.5; 13, Section 1.3]), we introduce biorthogonal wavelets and refinable functions whose masks are generalized Walsh polynomials. Note that for each $p \geq 2$ the generalized Walsh functions under consideration are characters of the corresponding Vilenkin group.

Let G be a locally compact abelian group consisting of sequences of the form

$$x = (x_j) = (\ldots, 0, 0, x_k, x_{k+1}, x_{k+2}, \ldots),$$

where $x_j \in \{0, 1, \ldots, p-1\}$ for $j \in \mathbb{Z}$ and $x_j = 0$ for $j < k = k(x)$. The group operation on G is denoted by $\oplus$ and defined as coordinatewise addition modulo p:

$$(z_j) = (x_j) \oplus (y_j) \quad \Leftrightarrow \quad z_j = x_j + y_j \ (\mathrm{mod}\, p) \quad \text{for} \quad j \in \mathbb{Z};$$

the topology on G is determined by the basis of neighborhoods of zero

$$U_l = \big\{(x_j) \in G \mid x_j = 0 \text{ for } j \leq l\big\}, \qquad l \in \mathbb{Z}.$$

The group G is known as the *Vilenkin group* (see, e.g., [2, p. 511]). We denote the inverse operation of $\oplus$ by $\ominus$ (so that $x \ominus x = \theta$, where θ is the zero sequence). Define U as a subgroup of G with the set of elements U_0.

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The Lebesgue spaces $L^q(G)$ with $1 \le q \le \infty$ are considered with respect to the Haar measure μ defined on the Borel subsets of G and normalized by the condition $\mu(U) = 1$. By $(\cdot\,,\cdot)$ and $\|\cdot\|$ we denote the inner product and the norm on $L^2(G)$.

The dual group of G is denoted by G^* and consists of sequences of the form

$$\omega = (\omega_j) = (\dots, 0, 0, \omega_k, \omega_{k+1}, \omega_{k+2}, \dots),$$

where $\omega_j \in \{0, 1, \dots, p-1\}$ for $j \in \mathbb{Z}$ and $\omega_j = 0$ for $j < k = k(\omega)$. Addition and subtraction, neighborhoods of zero $\{U_l^*\}$, and the Haar measure μ^* are defined for G^* in the same way as for G. Each character of the group G can be represented as

$$\chi(x, \omega) = \exp\left(\frac{2\pi i}{p} \sum_{j\in\mathbb{Z}} x_j \omega_{1-j}\right), \qquad x \in G,$$

for some $\omega \in G^*$. Consider the discrete subgroup $H = \{(x_j) \in G \mid x_j = 0 \text{ for } j > 0\}$ in G and define an automorphism $A \in \operatorname{Aut} G$ as $(Ax)_j = x_{j+1}$. It is easy to see that the quotient group $H/A(H)$ contains p elements, and the annihilator $H^\perp$ of H consists of those sequences $(\omega_j) \in G^*$ in which $\omega_j = 0$ for $j > 0$.

Consider the mapping $\lambda\colon G \to [0, +\infty)$ defined by

$$\lambda(x) = \sum_{j\in\mathbb{Z}} x_j p^{-j}, \qquad x = (x_j) \in G.$$

The image of the subgroup H under λ is the set of nonnegative integers: $\lambda(H) = \mathbb{Z}_+$. For each $\alpha \in \mathbb{Z}_+$, let $h_{[\alpha]}$ denote the element of H such that $\lambda(h_{[\alpha]}) = \alpha$ (in particular, $h_{[0]} = \theta$). A mapping $\lambda^*\colon G^* \to [0, +\infty)$, an automorphism $B \in \operatorname{Aut} G^*$, a subgroup U^* in G^*, and elements $\omega_{[\alpha]}$ in $H^\perp$ are defined by analogy with λ, A, U, and $h_{[\alpha]}$, respectively. Note that $\chi(Ax, \omega) = \chi(x, B\omega)$ for all $x \in G$ and $\omega \in G^*$ (i.e., the automorphism B is conjugate to A).

Generalized Walsh functions for the group G can be defined as

$$W_\alpha(x) = \chi(x, \omega_{[\alpha]}), \qquad \alpha \in \mathbb{Z}_+, \quad x \in G.$$

These functions are continuous on G and satisfy the orthogonality relations

$$\int_U W_\alpha(x)\overline{W_\beta(x)}\, d\mu(x) = \delta_{\alpha,\beta}, \qquad \alpha, \beta \in \mathbb{Z}_+,$$

where $\delta_{\alpha,\beta}$ denotes the Kronecker delta. It is known that the system $\{W_\alpha\}$ is complete in $L^2(U)$. The corresponding system for the group G^* is defined by

$$W_\alpha^*(\omega) = \chi(h_{[\alpha]}, \omega), \qquad \alpha \in \mathbb{Z}_+, \quad \omega \in G^*.$$

The system $\{W_\alpha^*\}$ is an orthonormal basis in $L^2(U^*)$.

For each function $f \in L^1(G) \cap L^2(G)$, its Fourier transform $\widehat{f}$,

$$\widehat{f}(\omega) = \int_G f(x)\overline{\chi(x, \omega)}\, d\mu(x), \qquad \omega \in G,$$

belongs to the space $L^2(G)$. The Fourier operator

$$\mathcal{F}\colon\ L^1(G) \cap L^2(G) \to L^2(G), \qquad \mathcal{F}f = \widehat{f},$$

admits a standard extension to the whole space $L^2(G)$. For any $f, g \in L^2(G)$, Parseval's equality $(f,g) = (\widehat{f}, \widehat{g})$ holds.

We denote the support of a function $f \in L^2(G)$ by $\operatorname{supp} f$; it is defined as the minimal (with respect to inclusion) closed set such that on its complement f vanishes almost everywhere. The set of functions in $L^2(G)$ with compact support is denoted by $L^2_{\mathrm{c}}(G)$.

Definition 1. A function $\varphi \in L^2_{\mathrm{c}}(G)$ is called a *refinable function* if it satisfies an equation of the form

$$\varphi(x) = p \sum_{\alpha=0}^{p^n-1} a_\alpha \varphi(Ax \ominus h_{[\alpha]}), \qquad x \in G, \tag{1.1}$$

where a_α are complex coefficients.

We denote the characteristic function of a set $E \subset G$ by $\mathbf{1}_E$. If $a_0 = \ldots = a_{p-1} = 1/p$ and $a_\alpha = 0$ for all $\alpha \geq p$, then the function $\varphi = \mathbf{1}_{U_{n-1}}$ is a solution of equation (1.1). The corresponding orthogonal wavelets in $L^2(G)$ have the form

$$\psi^{(\nu)}(x) = \sum_{\alpha=0}^{p-1} \varepsilon_p^{\nu\alpha} \varphi(Ax \ominus h_{[\alpha]}), \qquad \nu = 1, \ldots, p-1,$$

where $\varepsilon_p = \exp(2\pi i/p)$ (cf. [14, Theorem 2; 15, Section 4]). Some other examples of refinable functions determining orthogonal wavelets in $L^2(G)$ are given in [6] and [10].

The functional equation (1.1) is known as the *refinement equation*. Applying the Fourier transform, we can write this equation as

$$\widehat{\varphi}(\omega) = m(B^{-1}\omega)\widehat{\varphi}(B^{-1}\omega), \tag{1.2}$$

where

$$m(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W^*_\alpha(\omega)} \tag{1.3}$$

is a generalized Walsh polynomial, which is called the *mask* of the refinable function φ.

The sets

$$U^*_{n,s} = B^{-n}(\omega_{[s]}) \oplus B^{-n}(U^*), \qquad 0 \leq s \leq p^n - 1, \tag{1.4}$$

are cosets of the subgroup $B^{-n}(U^*)$ in the group U^*. Each function $W^*_\alpha(\cdot)$ with $0 \leq \alpha \leq p^n - 1$ is constant on sets (1.4). The coefficients of the refinement equation (1.1) are related to the values b_s of mask (1.3) on the classes $U^*_{n,s}$ by the direct and inverse Vilenkin–Chrestenson discrete transforms:

$$a_\alpha = \frac{1}{p^n} \sum_{s=0}^{p^n-1} b_s W^*_\alpha(B^{-n}\omega_{[s]}), \qquad 0 \leq \alpha \leq p^n - 1, \tag{1.5}$$

$$b_s = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W^*_\alpha(B^{-n}\omega_{[s]})}, \qquad 0 \leq s \leq p^n - 1. \tag{1.6}$$

Parseval's formula for these transforms takes the form

$$\sum_{\alpha=0}^{p^n-1} a_\alpha \overline{\widetilde{a}_\alpha} = \frac{1}{p^n} \sum_{s=0}^{p^n-1} b_s \overline{\widetilde{b}_s}, \tag{1.7}$$

where a_α, b_s and $\widetilde{a}_\alpha$, $\widetilde{b}_s$ are arbitrary sets of numbers satisfying equalities (1.5) and (1.6). The algorithms for calculating the Vilenkin–Chrestenson discrete transforms are similar to the classical fast Fourier transform algorithms (see, e.g., [2, p. 463]).

Theorem A. *If a function $\varphi \in L^2_{\mathrm{c}}(G)$ satisfies equation* (1.1) *and $\widehat{\varphi}(\theta) = 1$, then*

$$\sum_{\alpha=0}^{p^n-1} a_\alpha = 1 \qquad \text{and} \qquad \operatorname{supp} \varphi \subset U_{1-n}.$$

In the space $L^2_{\mathrm{c}}(G)$, this solution of equation (1.1) *is unique, is given by*

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(B^{-j}\omega),$$

and satisfies the following conditions:

(1) $\widehat{\varphi}(h^*) = 0$ *for all* $h^* \in H^\perp \setminus \{\theta\}$ (*a modified Strang–Fix condition*);

(2) $\sum_{h\in H} \varphi(x \oplus h) = 1$ *for a.e.* $x \in G$ (*the partition-of-unity property*).

Definition 2. We say that a function $f \in L^2(G)$ is *stable* if there exist positive constants c_1 and c_2 such that

$$c_1 \Biggl(\sum_{\alpha=0}^{\infty} |a_\alpha|^2\Biggr)^{1/2} \le \Biggl\|\sum_{\alpha=0}^{\infty} a_\alpha f(\,\cdot \ominus h_{[\alpha]})\Biggr\| \le c_2 \Biggl(\sum_{\alpha=0}^{\infty} |a_\alpha|^2\Biggr)^{1/2}$$

for every sequence $\{a_\alpha\}$ from ℓ^2. In other words, a function f is stable in $L^2(G)$ if the functions $f(\,\cdot \ominus h)$ with $h \in H$ form a Riesz system in $L^2(G)$.

We say that a function $g\colon G^* \to \mathbb{C}$ has a *periodic zero* at a point $\omega \in G^*$ if $g(\omega \oplus h^*) = 0$ for all $h^* \in H^\perp$. The following theorem characterizes compactly supported stable functions in $L^2(G)$.

Theorem B. *For any function $f \in L^2_{\mathrm{c}}(G)$, the following conditions are equivalent*:

(a) *the function f is stable in $L^2(G)$*;

(b) *the system $\{f(\,\cdot \ominus h) \mid h \in H\}$ is linearly independent*;

(c) *the Walsh transform of f has no periodic zeros.*

Theorems A and B were proved in [10] and [16] (their $L^2(\mathbb{R}_+)$ analogs for $p = 2$ can be found in [9]).

Definition 3. A family of closed subspaces $V_j \subset L^2(G)$, $j \in \mathbb{Z}$, is called a *multiresolution analysis* (or, briefly, an MRA) in $L^2(G)$ if the following conditions hold:

(i) $V_j \subset V_{j+1}$ for $j \in \mathbb{Z}$;

(ii) $\overline{\bigcup V_j} = L^2(G)$ and $\bigcap V_j = \{0\}$;

(iii) $f(\cdot) \in V_j \;\Leftrightarrow\; f(A\,\cdot) \in V_{j+1}$ for $j \in \mathbb{Z}$;

(iv) $f(\cdot) \in V_0 \;\Rightarrow\; f(\,\cdot \ominus h) \in V_0$ for $h \in H$;

(v) there exists a function $\varphi \in L^2(G)$ such that the system $\{\varphi(\,\cdot \ominus h) \mid h \in H\}$ is a Riesz basis in V_0.

Given a function $f \in L^2(G)$, we set

$$f_{j,h}(x) = p^{j/2} f(A^j x \ominus h), \qquad j \in \mathbb{Z}, \quad h \in H.$$

Definition 4. We say that a function φ *generates an MRA in* $L^2(G)$ if, first, the family $\{\varphi(\,\cdot\ominus h) \mid h \in H\}$ is a Riesz system in $L^2(G)$ and, second, the closed subspaces $V_j = \overline{\operatorname{span}}\{\varphi_{j,h} \mid h \in H\}$ with $j \in \mathbb{Z}$ form an MRA in $L^2(G)$.

In [8, 10], for arbitrary $p, n \in \mathbb{N}$ with $p \geq 2$, we found coefficients a_α such that the refinement equation (1.1) has a solution $\varphi \in L^2_{\mathrm{c}}(G)$ in the form of a lacunary series in generalized Walsh functions that generates an MRA in $L^2(G)$. Moreover, for each refinable function φ generating an MRA in $L^2(G)$, we can construct *orthogonal wavelets* $\psi^{(1)}, \dots, \psi^{(p-1)}$ such that the functions

$$\psi^{(\nu)}_{j,h}(x) = p^{j/2}\psi^{(\nu)}(A^j x \ominus h), \qquad 1 \leq \nu \leq p-1, \quad j \in \mathbb{Z}, \quad h \in H,$$

form an orthonormal basis in $L^2(G)$.

Let us expand a refinable function in a lacunary Walsh series. Let $l \in \{0, 1, \dots, p-1\}$, and let δ_l denote the sequence $\omega = (\omega_j)$ in which $\omega_1 = l$ and $\omega_j = 0$ for $j \neq 1$ (in particular, $\delta_0 = \theta$). It is easy to see that

$$\bigl\{\omega \in U^* \mid \chi(x, \omega) = 1 \text{ for } x \in A(H)\bigr\} = \{\delta_0, \delta_1, \dots, \delta_{p-1}\};$$

i.e., the set of sequences δ_l is an annihilator of the subgroup $A(H)$ in H. Note that $W^*_\alpha(\delta_l) = \varepsilon_p^{\alpha l}$ and $\delta_l = B^{-n}\omega_{[lp^{n-1}]}$ for $0 \leq l \leq p-1$.

Suppose that a function $\varphi \in L^2_{\mathrm{c}}(G)$ is a solution of the refinement equation (1.1) and its mask satisfies the conditions

$$m(\theta) = 1 \qquad \text{and} \qquad \sum_{l=0}^{p-1} |m(\omega \oplus \delta_l)|^2 = 1, \quad \omega \in G^*.$$

Then, as shown in [8], we have

$$\varphi(x) = \frac{1}{p^{n-1}}\,\mathbf{1}_U(A^{1-n}x)\left(1 + \sum_{l \in \mathbb{N}(p,n)} c_l[m] W_l(A^{1-n}x)\right), \qquad x \in G, \tag{1.8}$$

where $\mathbb{N}(p,n)$ and $c_l[m]$ are defined as follows. Let us represent each $l \in \mathbb{N}$ in the form of a p-ary expansion

$$l = \sum_{j=0}^{k} \mu_j p^j, \qquad \mu_j \in \{0, 1, \dots, p-1\}, \quad \mu_k \neq 0, \quad k = k(l) \in \mathbb{Z}_+, \tag{1.9}$$

and denote the set of all positive integers $l \geq p^{n-1}$ for which the ordered sets $(\mu_j, \mu_{j+1}, \dots, \mu_{j+n-1})$ of the coefficients of (1.9) do not contain

$$(0, 0, \dots, 0, 1),\ (0, 0, \dots, 0, 2),\ \dots,\ (0, 0, \dots, 0, p-1)$$

by $\mathbb{N}_0(p,n)$. Then $\mathbb{N}(p,n) = \{1, 2, \dots, p^{n-1}-1\} \cup \mathbb{N}_0(p,n)$. Let

$$\gamma(i_1, i_2, \dots, i_n) = b_s, \qquad s = i_1 p^0 + i_2 p^1 + \dots + i_n p^{n-1}, \quad i_j \in \{0, 1, \dots, p-1\},$$

where b_s are defined by (1.6). Then

$$c_l[m] = \gamma(\mu_0, 0, 0, \dots, 0, 0) \qquad \text{if} \quad k(l) = 0,$$

$$c_l[m] = \gamma(\mu_1, 0, 0, \dots, 0, 0)\gamma(\mu_0, \mu_1, 0, \dots, 0, 0) \qquad \text{if} \quad k(l) = 1,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$c_l[m] = \gamma(\mu_k, 0, 0, \dots, 0, 0)\gamma(\mu_{k-1}, \mu_k, 0, \dots, 0, 0)\dots\gamma(\mu_0, \mu_1, \mu_2, \dots, \mu_{n-2}, \mu_{n-1})$$
$$\text{if} \quad k(l) \geq n-1.$$

Note that in the last product the subscripts of each factor starting with the second are obtained by shifting those of the preceding factor by one position to the right and placing one new digit from the p-ary decomposition (1.9) at the vacant first position.

2. CONSTRUCTION OF BIORTHOGONAL WAVELETS ON THE VILENKIN GROUP

Suppose given two refinable functions φ and $\widetilde{\varphi}$ with masks

$$m(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W_\alpha^*(\omega)} \qquad \text{and} \qquad \widetilde{m}(\omega) = \sum_{\alpha=0}^{p^{\widetilde{n}}-1} \widetilde{a}_\alpha \overline{W_\alpha^*(\omega)}, \tag{2.1}$$

respectively. We are interested in the following questions:

(i) When do the H-shifts of the functions φ and $\widetilde{\varphi}$ form a biorthonormal system in $L^2(G)$?

(ii) How can one construct biorthogonal bases in $L^2(G)$ from masks (2.1)?

Proposition 1. *Let $\varphi, \widetilde{\varphi} \in L^2(G)$. The systems $\{\varphi(\,\cdot \ominus h) \mid h \in H\}$ and $\{\widetilde{\varphi}(\,\cdot \ominus h) \mid h \in H\}$ are biorthonormal in $L^2(G)$ if and only if*

$$\sum_{h^* \in H^\perp} \widehat{\varphi}(\omega \oplus h^*)\, \overline{\widehat{\widetilde{\varphi}}(\omega \oplus h^*)} = 1 \qquad \text{for a.e. } \omega \in G^*.$$

Proposition 2. *Let φ and $\widetilde{\varphi}$ be refinable functions with masks m and $\widetilde{m}$, respectively. If the systems $\{\varphi(\,\cdot \ominus h) \mid h \in H\}$ and $\{\widetilde{\varphi}(\,\cdot \ominus h) \mid h \in H\}$ are biorthonormal in $L^2(G)$, then*

$$\sum_{l=0}^{p-1} m(\omega \oplus \delta_l)\, \overline{\widetilde{m}(\omega \oplus \delta_l)} = 1 \qquad \text{for all } \omega \in G^*. \tag{2.2}$$

Analogs of Propositions 1 and 2 were proved in [8, Section 3] and [13, Section 1.2].

Given masks (2.1), we set $m^*(\omega) = m(\omega)\overline{\widetilde{m}(\omega)}$ and $N = \max\{n, \widetilde{n}\}$. Condition (2.2) is then written in the form

$$\sum_{l=0}^{p-1} m^*(\omega \oplus \delta_l) = 1, \qquad \omega \in G^*,$$

and is equivalent to the equalities

$$\sum_{\nu=0}^{p-1} b^{(N)}_{l+\nu p^{N-1}}\, \overline{\widetilde{b}}^{(N)}_{l+\nu p^{N-1}} = 1, \qquad 0 \le l \le p^{N-1} - 1, \tag{2.3}$$

where $b_l^{(N)} = m(B^{-N}\omega_{[l]})$ and $\widetilde{b}_l^{(N)} = \widetilde{m}(B^{-N}\omega_{[l]})$. Let $\oplus_p$ and $\ominus_p$ denote, respectively, addition and subtraction of integers modulo p. The equality $W_\alpha^*(\omega)W_\beta^*(\omega) = W^*_{\alpha\oplus_p\beta}(\omega)$ implies

$$m^*(\omega) = \sum_{\alpha=0}^{p^n-1} \sum_{\beta=0}^{p^{\widetilde{n}}-1} a_\alpha \overline{\widetilde{a}}_\beta \overline{W^*_{\alpha\oplus_p\beta}(\omega)}.$$

Setting $a_\alpha = \widetilde{a}_\beta = 0$ for $\alpha \ge p^n$ and $\beta \ge p^{\widetilde{n}}$, we obtain

$$m^*(\omega) = \sum_{\alpha=0}^{p^N-1} a_\alpha^* \overline{W_\alpha^*(\omega)}, \qquad a_\alpha^* = \sum_{\gamma=0}^{p^N-1} a_\gamma \overline{\widetilde{a}}_{\gamma\ominus_p\alpha}.$$

Consider the function φ^* defined by

$$\varphi^*(x) = \int\limits_G \varphi(t \oplus x)\overline{\widetilde{\varphi}(t)}\, d\mu(t).$$

The Fourier transform of this function is related to those of φ and $\widetilde{\varphi}$ by $\widehat{\varphi}^*(\omega) = \widehat{\varphi}(\omega)\overline{\widehat{\widetilde{\varphi}}(\omega)}$. Moreover, φ^* is a refinable function satisfying the equation

$$\varphi^*(x) = p \sum_{\alpha=0}^{p^N-1} a_\alpha^* \varphi^*(Ax \ominus \alpha), \qquad x \in G.$$

Thus, the polynomial m^* is the mask of the function φ^*.

For $M \subset U^*$, let

$$S_p M = \bigcup_{l=0}^{p-1} \bigl\{B^{-1}\omega_{[l]} + B^{-1}(\omega) \mid \omega \in M\bigr\}.$$

Suppose that M either coincides with one of the sets $U^*_{n-1,\,s}$, $1 \le s \le p^{n-1} - 1$, or is the union of some of these sets. Such a set M is said to be *blocking* for a mask m if $m(\omega) = 0$ for all $\omega \in S_p M \setminus M$. According to this definition, if M is a blocking set for m, then $M \cap U^*_{n-1,\,0} = \varnothing$. Moreover, each mask may have only finitely many blocking sets.

Proposition 3. *If one of the masks m, $\widetilde{m}$, and m^* has a blocking set, then the systems $\{\varphi(\,\cdot\ominus h) \mid h \in H\}$ and $\{\widetilde{\varphi}(\,\cdot \ominus h) \mid h \in H\}$ are not biorthonormal in $L^2(G)$.*

This proposition is proved by using Theorem B (the case of $m = \widetilde{m}$ was considered in detail in [10]).

Let E be a compact set in G^*. The set E is said to be *congruent to U^* modulo $H^\perp$* if $\mu^*(E) = 1$ and, for any $\omega \in E$, there exists an $h^* \in H^\perp$ such that $\omega \oplus h^* \in U^*$. The following analog of the well-known Cohen's criterion (see, e.g., [13, Theorem 2.5.6]) is valid.

Theorem 1. *If φ and $\widetilde{\varphi}$ are refinable functions whose masks m and $\widetilde{m}$ satisfy condition* (2.2) *and $\widehat{\varphi}(\theta) = \widehat{\widetilde{\varphi}}(\theta) = 1$, then the following conditions are equivalent*:

(a) *the systems $\{\varphi(\,\cdot \ominus h) \mid h \in H\}$ and $\{\widetilde{\varphi}(\,\cdot \ominus h) \mid h \in H\}$ are biorthonormal in $L^2(G)$;*

(b) *there exists a set E that is congruent to U^* modulo $H^\perp$, contains a neighborhood of zero in G^*, and is such that*

$$\inf_{j \in \mathbb{N}} \inf_{\omega \in E} |m(B^{-j}\omega)| > 0 \qquad \textit{and} \qquad \inf_{j \in \mathbb{N}} \inf_{\omega \in E} |\widetilde{m}(B^{-j}\omega)| > 0. \tag{2.4}$$

For $\varphi = \widetilde{\varphi}$, this theorem was proved in [10]. When constructing biorthogonal wavelet systems, one of the main questions is whether the functions φ and $\widetilde{\varphi}$ belong to the class L^2. In some cases, this question can be answered by using the joint spectral radius of special linear operators defined for the corresponding refinement equations (see, e.g., Theorem A.6.5 in [13]).

Let $r = p^{n-1}$. Recall that the joint spectral radius of $r \times r$ complex matrices $A_0, A_1, \dots, A_{p-1}$ is defined as

$$\widehat{\rho}(A_0, A_1, \dots, A_{p-1}) := \lim_{k \to \infty} \max\Bigl\{\|A_{d_1} A_{d_2} \dots A_{d_k}\|^{1/k} \colon\ d_j \in \{0, 1, \dots, p-1\},\ 1 \le j \le k\Bigr\},$$

where $\|\cdot\|$ is an arbitrary norm on $\mathbb{C}^{r \times r}$. In the case of $A_0 = A_1 = \ldots = A_{p-1}$, $\widehat{\rho}(A_0, A_1, \dots, A_{p-1})$ coincides with the spectral radius $\rho(A_0)$. The joint spectral radius of finite-dimensional linear operators L_0, $L_1, \dots, L_{p-1}$ is defined as the joint spectral radius of their matrices in an arbitrary fixed basis of the corresponding linear space.

Given a refinement equation of the form (1.1), we set $c_\alpha = pa_\alpha$ and define $r\times r$ matrices $T_0, T_1, \dots, T_{p-1}$ by

$$(T_0)_{i,j} = c_{(pi-p)\ominus_p(j-1)}, \quad (T_1)_{i,j} = c_{(pi-p+1)\ominus_p(j-1)}, \quad \dots, \quad (T_{p-1})_{i,j} = c_{(pi-1)\ominus_p(j-1)}$$

for $i,j\in\{1,2,\dots,r\}$. Consider the subspace

$$V := \bigl\{u=(u_1,\dots,u_r)^{\mathrm{t}} \mid u_1+\dots+u_r=0\bigr\}$$

and denote by $L_0, L_1, \dots, L_{p-1}$ the restrictions to V of the linear operators defined on the whole space $\mathbb{C}^r$ by the matrices $T_0, T_1, \dots, T_{p-1}$, respectively.

Proposition 4. *If the mask m of the refinement equation* (1.1) *satisfies the conditions*

$$m(\theta)=1, \qquad m(\delta_1)=m(\delta_2)=\dots=m(\delta_{p-1})=0$$

and $\widehat{\rho}[m] := \widehat{\rho}(L_0, L_1, \dots, L_{p-1}) < 1$, then the function φ defined by (1.8) *satisfies equation* (1.1) *and is continuous on G.*

This proposition can be proved by analogy with similar results in [8, Sections 2 and 4]; it turns out that the series in (1.8) converges uniformly on G (see also Section 5 in [9]). It is easy to see that, under the conditions of Proposition 4, the function φ has compact support and belongs to the space $L^2(G)$. Moreover, in this case

$$\sum_{\alpha=0}^{p^{n-1}-1} a_{p\alpha} = \sum_{\alpha=0}^{p^{n-1}-1} a_{p\alpha+1} = \dots = \sum_{\alpha=0}^{p^{n-1}-1} a_{p\alpha+p-1} = \frac{1}{p}.$$

In the graphic illustrations given below, the functions Φ and $\widetilde{\Phi}$ are defined on $[0,+\infty)$ and related to the refinable functions φ and $\widetilde{\varphi}$ by the equalities $\varphi(x) = \Phi[\lambda(x)]$ and $\widetilde{\varphi}(x) = \widetilde{\Phi}[\lambda(x)]$ for almost every $x\in G$ (obviously, the mapping $\lambda\colon G\to[0,+\infty)$ defined in Section 1 is invertible almost everywhere). Note also that in Examples 1–3 expansions of the form (1.8) are derived from the formulas

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(B^{-j}\omega) \qquad \text{and} \qquad \widehat{\widetilde{\varphi}}(\omega) = \prod_{j=1}^{\infty} \widetilde{m}(B^{-j}\omega) \tag{2.5}$$

by means of the Fourier transform.

Example 1. Let $p=2$, $n=\widetilde{n}=2$, and the masks of refinable functions φ and $\widetilde{\varphi}$ have the form

$$m(\omega) = \begin{cases} 1, & \omega\in U^*_{2,0},\\ a, & \omega\in U^*_{2,1},\\ 0, & \omega\in U^*_{2,2},\\ b, & \omega\in U^*_{2,3}, \end{cases} \qquad \text{and} \qquad \widetilde{m}(\omega) = \begin{cases} 1, & \omega\in U^*_{2,0},\\ \widetilde{a}, & \omega\in U^*_{2,1},\\ 0, & \omega\in U^*_{2,2},\\ \widetilde{b}, & \omega\in U^*_{2,3}, \end{cases} \tag{2.6}$$

where $a\overline{\widetilde{a}} + b\overline{\widetilde{b}} = 1$. If $a=0$ or $\widetilde{a}=0$, then $U^*_{1,1}$ is a blocking set for m or $\widetilde{m}$; otherwise, there are no blocking sets for the masks m, $\widetilde{m}$, and m^*. Suppose in addition that $|b|<1$ and $|\widetilde{b}|<1$. Then $a\widetilde{a}\neq 0$, there are no blocking sets, and condition (2.4) holds for $E=U^*$. Moreover, as in the orthogonal case considered in [4], we have

$$\varphi(x) = \frac{1}{2}\,\mathbf{1}_U(A^{-1}x)\Biggl(1 + a\sum_{j=0}^{\infty} b^j W_{2^{j+1}-1}(A^{-1}x)\Biggr),$$

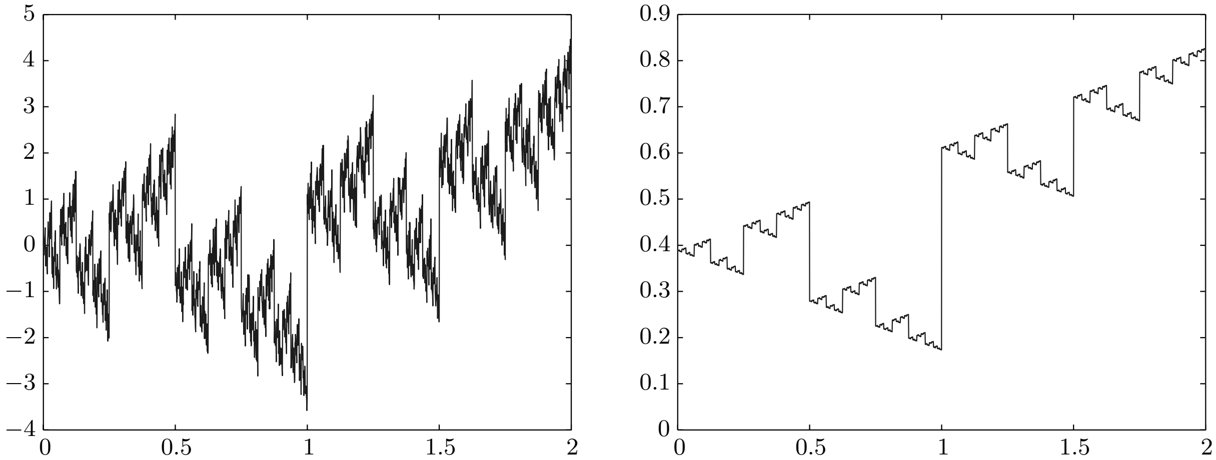

Fig. 1. The functions Φ (left) and $\widetilde{\Phi}$ (right) from Example 1.

$\widetilde{\varphi}$ admits a similar expansion, and both functions φ and $\widetilde{\varphi}$ are continuous on G. Indeed, the required expansions for φ and $\widetilde{\varphi}$ are obtained directly from (2.5) and (2.6), and it is seen from Example 4.3 in [8] and Remark 3 in [9] that $\widehat{\rho}\,[m] = |b|$ and $\widehat{\rho}\,[\widetilde{m}] = |\widetilde{b}|$; therefore, Proposition 4 applies. Thus, if $a\overline{\widetilde{a}} + b\overline{\widetilde{b}} = 1$, $|b| < 1$, and $|\widetilde{b}| < 1$, then the H-shifts of the refinable functions φ and $\widetilde{\varphi}$ form a biorthonormal system in $L^2(G)$. The graphs of the functions Φ and $\widetilde{\Phi}$ for

$$a = -1.835358, \qquad b = -0.792570, \qquad \widetilde{a} = -0.332874, \qquad \widetilde{b} = -0.490884$$

are shown in Fig. 1.

Example 2. Let $p = 2$, $n = 3$, $\widetilde{n} = 2$, the mask $\widetilde{m}$ be the same as in (2.6), and the mask m be given by

$$\begin{aligned} m(\omega) &= 1 \quad \text{for } \omega \in U^*_{3,0}, & m(\omega) &= 1 \quad \text{for } \omega \in U^*_{3,1},\\ m(\omega) &= b \quad \text{for } \omega \in U^*_{3,2}, & m(\omega) &= c \quad \text{for } \omega \in U^*_{3,3},\\ m(\omega) &= 0 \quad \text{for } \omega \in U^*_{3,4}, & m(\omega) &= 0 \quad \text{for } \omega \in U^*_{3,5},\\ m(\omega) &= \beta \quad \text{for } \omega \in U^*_{3,6}, & m(\omega) &= \gamma \quad \text{for } \omega \in U^*_{3,7}. \end{aligned}$$

Then, the class $U^*_{1,1}$ is a blocking set for m, $\widetilde{m}$, and m^* if (1) $b = c = 0$, (2) $\widetilde{a} = 0$, and (3) $b\widetilde{a} = c\widetilde{a} = 0$, respectively. Moreover, $U^*_{2,3}$ is a blocking set for m and m^* if $c = 0$ and $c\widetilde{a} = 0$, respectively. There are no other blocking sets for m, $\widetilde{m}$, and m^*. Note that if $\widetilde{a} = 0$ and $bc \neq 0$, then $U^*_{2,3}$ is a blocking set for m^*, but it is not a blocking set for m and $\widetilde{m}$. In accordance with (2.3), suppose that

$$b\overline{\widetilde{a}} + \beta\overline{\widetilde{b}} = c\overline{\widetilde{a}} + \gamma\overline{\widetilde{b}} = 1$$

(in particular, for $b = 0$ and $c = 1$, we have $\beta = 1/\overline{\widetilde{b}}$ and $\gamma = (1 - \widetilde{a})/\overline{\widetilde{b}}\,)$. Then blocking sets for m, $\widetilde{m}$, and m^* exist only if $\widetilde{a} = 0$ or $c = 0$. In the case of $c\widetilde{a} \neq 0$, condition (2.4) holds for $E = U^*_{2,0} \cup U^*_{3,3} \cup U^*_{3,6}$. According to Examples 4.3 and 4.4 from [8], we have $\widehat{\rho}\,[m] = |\gamma|$ and $\widehat{\rho}\,[\widetilde{m}] = |\widetilde{b}|$. Thus, if $c\widetilde{a} \neq 0$, $|\gamma| < 1$, and $|\widetilde{b}| < 1$, then the H-shifts of the refinable functions φ and $\widetilde{\varphi}$ form a biorthonormal system in $L^2(G)$. The graphs of the functions Φ and $\widetilde{\Phi}$ for

$$b = 1.201260, \qquad c = 1.166263, \qquad \beta = -0.367477, \qquad \gamma = -0.477955,$$

$$\widetilde{a} = 0.758916, \qquad \widetilde{b} = -0.240408$$

are shown in Fig. 2.

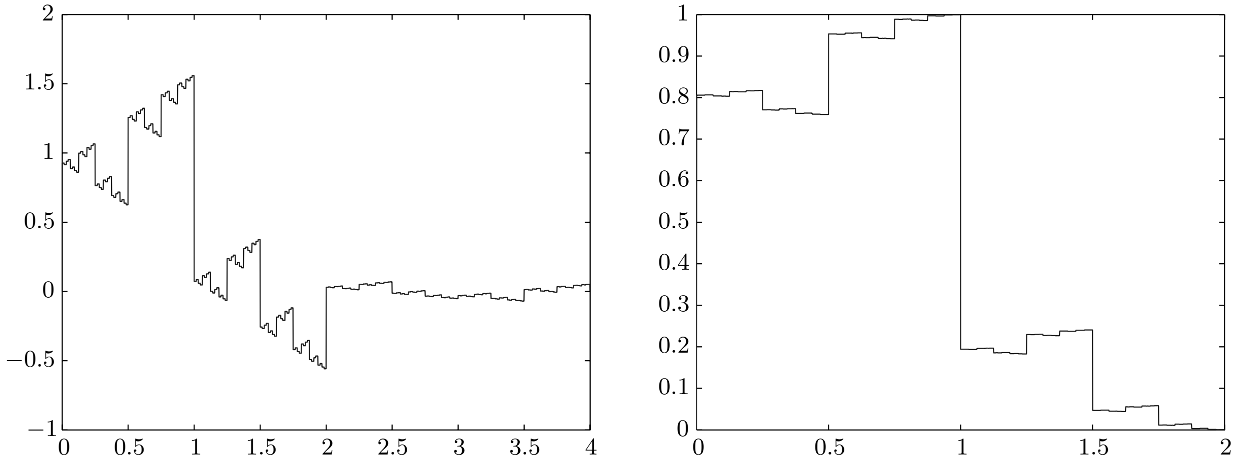

Fig. 2. The functions Φ (left) and $\widetilde{\Phi}$ (right) from Example 2.

Example 3. Let $p = 3$, $n = \widetilde{n} = 2$, and the masks m and $\widetilde{m}$ take the value 1 on $U^*_{2,0}$, vanish on $U^*_{2,3} \cup U^*_{2,6}$, and be defined on the remaining part of the subgroup U^* by the equalities

$$m(\omega) = a \ \text{ and } \ \widetilde{m}(\omega) = \widetilde{a} \quad \text{for } \ \omega \in U^*_{2,1}, \qquad m(\omega) = \alpha \ \text{ and } \ \widetilde{m}(\omega) = \widetilde{\alpha} \quad \text{for } \ \omega \in U^*_{2,2},$$

$$m(\omega) = b \ \text{ and } \ \widetilde{m}(\omega) = \widetilde{b} \quad \text{for } \ \omega \in U^*_{2,4}, \qquad m(\omega) = \beta \ \text{ and } \ \widetilde{m}(\omega) = \widetilde{\beta} \quad \text{for } \ \omega \in U^*_{2,5},$$

$$m(\omega) = c \ \text{ and } \ \widetilde{m}(\omega) = \widetilde{c} \quad \text{for } \ \omega \in U^*_{2,7}, \qquad m(\omega) = \gamma \ \text{ and } \ \widetilde{m}(\omega) = \widetilde{\gamma} \quad \text{for } \ \omega \in U^*_{2,8},$$

where the parameters satisfy the condition

$$a\overline{\widetilde{a}} + b\overline{\widetilde{b}} + c\overline{\widetilde{c}} = \alpha\overline{\widetilde{\alpha}} + \beta\overline{\widetilde{\beta}} + \gamma\overline{\widetilde{\gamma}} = 1.$$

Then, in the cases $a\overline{\widetilde{a}} = \alpha\overline{\widetilde{\alpha}} = 0$, $a\overline{\widetilde{a}} = c\overline{\widetilde{c}} = 0$, and $\alpha\overline{\widetilde{\alpha}} = \beta\overline{\widetilde{\beta}} = 0$, blocking sets for the mask m^* are $U^*_{1,1} \cup U^*_{1,2}$, $U^*_{1,1}$, and $U^*_{1,2}$, respectively. Condition (2.4) holds in the following three cases:

(1) $a\widetilde{a} \neq 0$, $\alpha\widetilde{\alpha} \neq 0$, and $E = U^*$;

(2) $a\widetilde{a} \neq 0$, $\beta\widetilde{\beta} \neq 0$, and $E = U^*_{1,0} \cup U^*_{1,1} \cup U^*_{1,5}$;

(3) $c\widetilde{c} \neq 0$, $\alpha\widetilde{\alpha} \neq 0$, and $E = U^*_{1,0} \cup U^*_{1,2} \cup U^*_{1,7}$.

Thus, if $\widehat{\rho}\,[m] < 1$ and $\widehat{\rho}\,[\widetilde{m}] < 1$ (this condition can be verified numerically for specific parameter values), then, in the above three cases, the H-shifts of the refinable functions φ and $\widetilde{\varphi}$ form a biorthonormal system in $L^2(G)$. Figures 3 and 4 show the real and imaginary parts of the functions Φ and $\widetilde{\Phi}$ for the following parameter values:

$$a = 0.9, \quad b = -0.295272, \quad c = 0.403503, \quad \alpha = 0.9, \quad \beta = 0.478760, \quad \gamma = 0.144181,$$

$$\widetilde{a} = 0.9, \quad \widetilde{b} = 0.037839, \quad \widetilde{c} = 0.498566, \quad \widetilde{\alpha} = 0.9, \quad \widetilde{\beta} = 0.270151, \quad \widetilde{\gamma} = 0.420735.$$

Note that for these values of the parameters $\widehat{\rho}\,[m] < 0.57$ and $\widehat{\rho}\,[\widetilde{m}] < 0.65$.

Definition 5. Let $\{V_j\}$ and $\{\widetilde{V}_j\}$ be two MRAs in $L^2(G)$. We say that functions $\psi^{(\nu)} \in V_1$ and $\widetilde{\psi}^{(\nu)} \in \widetilde{V}_1$, $\nu = 1, \dots, p-1$, form a *biorthogonal wavelet set* with respect to the pair $\{V_j\}$, $\{\widetilde{V}_j\}$ if $\psi^{(\nu)} \perp \widetilde{V}_0$ and $\widetilde{\psi}^{(\nu)} \perp V_0$ for all $\nu = 1, \dots, p-1$ and

$$\big(\psi^{(\nu)}(\,\cdot \oplus h_{[\alpha]}), \widetilde{\psi}^{(\varkappa)}(\,\cdot \oplus h_{[\beta]})\big) = \delta_{\nu,\varkappa}\,\delta_{\alpha,\beta}, \qquad \nu, \varkappa \in \{1, \dots, p-1\}, \quad \alpha, \beta \in \mathbb{Z}_+.$$

As usual, by $\mathcal{M}^*$ we denote the adjoint matrix of $\mathcal{M}$. The identity matrix of order p is denoted by E_p.

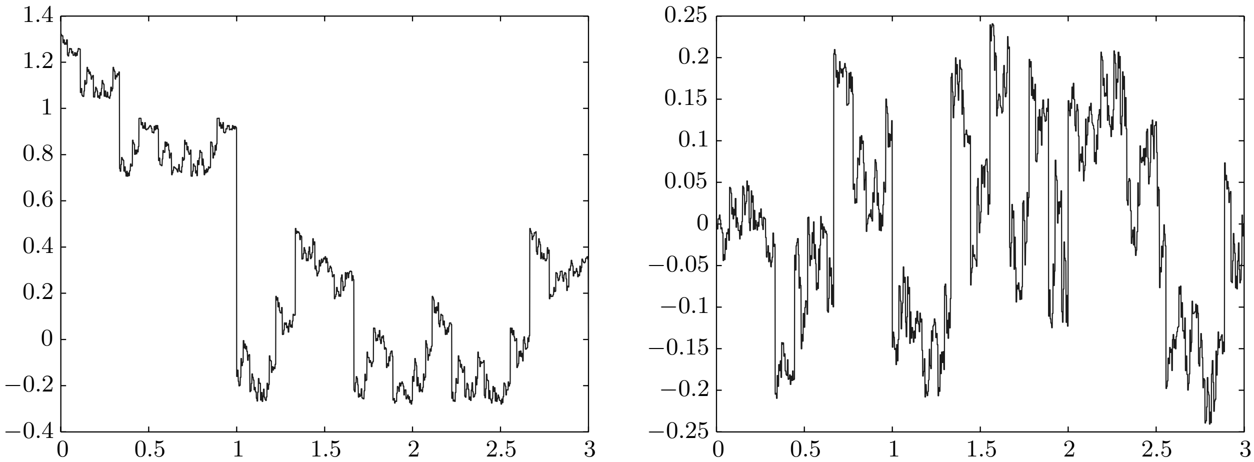

Fig. 3. The functions $\operatorname{Re}\Phi$ (left) and $\operatorname{Im}\Phi$ (right) from Example 3.

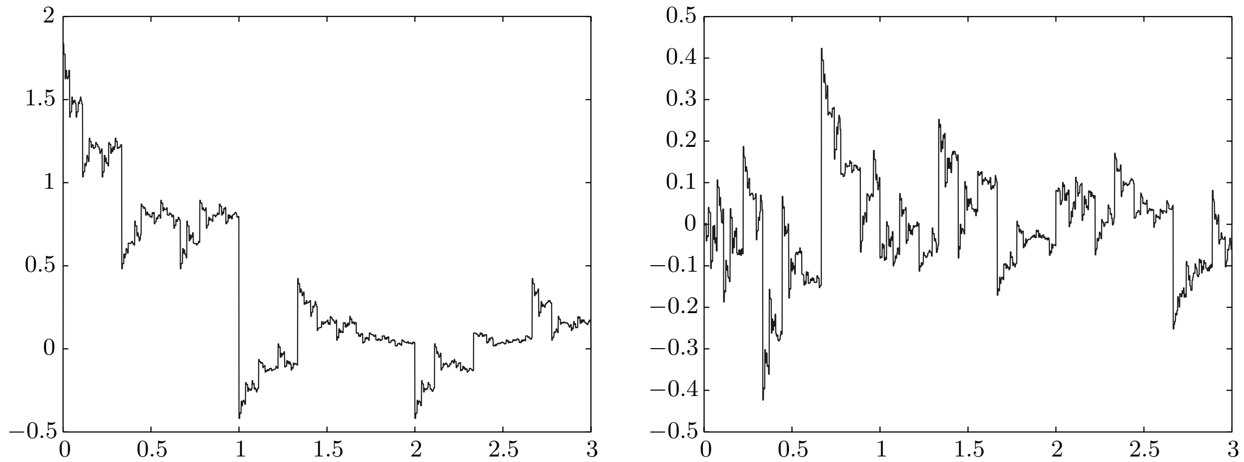

Fig. 4. The functions $\operatorname{Re}\widetilde{\Phi}$ (left) and $\operatorname{Im}\widetilde{\Phi}$ (right) from Example 3.

Theorem 2. *Suppose that $\{V_j\}$ and $\{\widetilde{V}_j\}$ are MRAs generated by refinable functions φ and $\widetilde{\varphi}$ with masks $m = m_0$ and $\widetilde{m} = \widetilde{m}_0$, respectively, and the systems $\{\varphi(\,\cdot\ominus h) \mid h \in H\}$ and $\{\widetilde{\varphi}(\,\cdot\ominus h) \mid h \in H\}$ are biorthonormal. If the matrices*

$$\mathcal{M} = \{m_\nu(\omega + \delta_k)\}_{\nu,k=0}^{p-1} \qquad \text{and} \qquad \widetilde{\mathcal{M}} = \{\widetilde{m}_\nu(\omega + \delta_k)\}_{\nu,k=0}^{p-1}, \tag{2.7}$$

where $m_\nu, \widetilde{m}_\nu \in L^2(U^*)$, satisfy the condition

$$\mathcal{M}\widetilde{\mathcal{M}}^* = E_p \tag{2.8}$$

for almost every $\omega \in U^*$, then the functions $\psi^{(\nu)}$ and $\widetilde{\psi}^{(\nu)}$, $\nu = 1, \dots, p-1$, defined by the equalities

$$\widehat{\psi}^{(\nu)}(\omega) = m_\nu(B^{-1}\omega)\widehat{\varphi}(B^{-1}\omega) \qquad \text{and} \qquad \widehat{\widetilde{\psi}^{(\nu)}}(\omega) = \widetilde{m}_\nu(B^{-1}\omega)\widehat{\widetilde{\varphi}}(B^{-1}\omega) \tag{2.9}$$

form a biorthogonal wavelet set with respect to the pair $\{V_j\}$, $\{\widetilde{V}_j\}$.

Analogs of Theorem 2 for the spaces $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}_+)$ were proved in [13, Ch. 2] and [17]. For $p = 2$, we can take

$$m_1(\omega) = -w_1(\omega)\overline{\widetilde{m}_0(\omega \oplus \delta_1)} \qquad \text{and} \qquad \widetilde{m}_1(\omega) = -w_1(\omega)\overline{m_0(\omega \oplus \delta_1)}. \tag{2.10}$$

Applying the inverse Fourier transform and setting $\psi = \psi^{(1)}$ and $\widetilde{\psi} = \widetilde{\psi}^{(1)}$, we deduce from (2.1), (2.9), and (2.10) that

$$\psi(x) = 2\sum_{\alpha=0}^{2^n-1}(-1)^\alpha\,\overline{\widetilde{a}}_{\alpha\oplus 1}\varphi(Ax\ominus h_{[\alpha]}) \qquad\text{and}\qquad \widetilde{\psi}(x) = 2\sum_{\alpha=0}^{2^{\tilde{n}}-1}(-1)^\alpha\,\overline{a}_{\alpha\oplus 1}\widetilde{\varphi}(Ax\ominus h_{[\alpha]}).$$

According to (2.1), the polynomials $m_0 = m$ and $\widetilde{m}_0 = \widetilde{m}$ satisfy the equalities

$$m_0(\omega) = \sum_{k=0}^{p-1}\overline{W_k^*(\omega)}A_{0k}(\overline{W_p^*(\omega)}) \qquad\text{and}\qquad \widetilde{m}_0(\omega) = \sum_{k=0}^{p-1}\overline{W_k^*(\omega)}\widetilde{A}_{0k}(\overline{W_p^*(\omega)}),$$

where

$$A_{0k}(z) = \sum_{l=0}^{p^{n-1}-1} a_{k+pl}z^l \qquad\text{and}\qquad \widetilde{A}_{0k}(z) = \sum_{l=0}^{p^{\tilde{n}-1}-1}\widetilde{a}_{k+pl}z^l, \qquad 0\le k\le p-1.$$

Let $\mathbb{T}$ be the unit circle of the complex plane $\mathbb{C}$. Choose the coefficients a_α and $\widetilde{a}_\alpha$ in (2.1) so that

$$p\sum_{k=0}^{p-1}A_{0k}(z)\overline{\widetilde{A}_{0k}(z)} = 1, \qquad z\in\mathbb{T}, \tag{2.11}$$

and

$$\sum_{k=0}^{p-1}A_{0k}(1) = \sum_{k=0}^{p-1}\widetilde{A}_{0k}(1) = 1 \tag{2.12}$$

(it is easy to see that $m_0(\theta) = \widetilde{m}_0(\theta) = 1$ in this case). Suppose that we have found algebraic polynomials $A_{\nu k}(z)$ and $\widetilde{A}_{\nu k}(z)$, $1\le \nu, k\le p-1$, such that

$$p\sum_{k=0}^{p-1}A_{\nu k}(z)\overline{\widetilde{A}_{\varkappa k}(z)} = \delta_{\nu,\varkappa}, \qquad 1\le\nu,\varkappa\le p-1, \quad z\in\mathbb{T}, \tag{2.13}$$

and

$$\sum_{k=0}^{p-1}A_{\nu k}(1) = \sum_{k=0}^{p-1}\widetilde{A}_{\nu k}(1) = 0, \qquad 1\le\nu\le p-1. \tag{2.14}$$

Then we set

$$m_\nu(\omega) = \sum_{k=0}^{p-1}\overline{W_k^*(\omega)}A_{\nu k}(\overline{W_p^*(\omega)}), \qquad \widetilde{m}_\nu(\omega) = \sum_{k=0}^{p-1}\overline{W_k^*(\omega)}\widetilde{A}_{\nu k}(\overline{W_p^*(\omega)}), \qquad 1\le\nu\le p-1,$$

and form the matrices

$$\mathcal{M} = \{m_\nu(\omega\oplus\delta_k)\}_{\nu,k=0}^{p-1}, \qquad \widetilde{\mathcal{M}} = \{\widetilde{m}_\nu(\omega\oplus\delta_k)\}_{\nu,k=0}^{p-1}, \qquad \mathcal{D}(\omega) = \{W_\nu^*(\omega\oplus\delta_k)\}_{\nu,k=0}^{p-1}.$$

It is easy to see that

$$\mathcal{M} = A(\overline{W_p^*(\omega)})\mathcal{D}(\omega) \qquad\text{and}\qquad \widetilde{\mathcal{M}} = \widetilde{A}(\overline{W_p^*(\omega)})\mathcal{D}(\omega), \tag{2.15}$$

where $A(z) = \{A_{\nu k}(z)\}_{\nu,k=0}^{p-1}$ and $\widetilde{A}(z) = \{\widetilde{A}_{\nu k}(z)\}_{\nu,k=0}^{p-1}$. Since the matrix $p^{-1/2}\mathcal{D}(\omega)$ is unitary, it follows from (2.15) that

$$\mathcal{M}\widetilde{\mathcal{M}}^* = pA(\overline{W_p^*(\omega)})\widetilde{A}^*(\overline{W_p^*(\omega)}), \tag{2.16}$$

or, in more detail,

$$\sum_{k=0}^{p-1} m_\nu(\omega \oplus \delta_k)\overline{\widetilde{m}_\varkappa(\omega \oplus \delta_k)} = p\sum_{k=0}^{p-1} A_{\nu k}(\overline{W_p^*(\omega)})\overline{\widetilde{A}_{\varkappa k}(\overline{W_p^*(\omega)})}, \qquad 1 \le \nu, \varkappa \le p-1.$$

Formulas (2.13), (2.14), and (2.16) show that matrices (2.15) satisfy condition (2.8); moreover, $m_\nu(\theta) = 0$ and $\widetilde{m}_\nu(\theta) = 0$ for $\nu = 1, \dots, p-1$.

Example 4. Let $p = 3$ and $n = \widetilde{n} = 2$. We define the coefficients of the masks $m = m_0$ and $\widetilde{m} = \widetilde{m}_0$ by means of (1.5) choosing the parameters $a, \alpha, \dots, \overline{\widetilde{\gamma}}$ in such a way that the systems $\{\varphi(\,\cdot \ominus h) \mid h \in H\}$ and $\{\widetilde{\varphi}(\,\cdot \ominus h) \mid h \in H\}$ are biorthonormal in $L^2(G)$ (see Example 3). Recall that

$$a\overline{\widetilde{a}} + b\overline{\widetilde{b}} + c\overline{\widetilde{c}} = \alpha\overline{\widetilde{\alpha}} + \beta\overline{\widetilde{\beta}} + \gamma\overline{\widetilde{\gamma}} = 1. \tag{2.17}$$

Since $b_0 = \widetilde{b}_0 = 1$, it follows from (1.7) and (2.17) that

$$\sum_{\alpha=0}^{8} a_\alpha \overline{\widetilde{a}_\alpha} = \frac{1}{3}.$$

Moreover, in the case under consideration,

$$A_{00}(z) = a_0 + a_3 z + a_6 z^2, \qquad A_{01}(z) = a_1 + a_4 z + a_7 z^2, \qquad A_{02}(z) = a_2 + a_5 z + a_8 z^2,$$

$$\widetilde{A}_{00}(z) = \widetilde{a}_0 + \widetilde{a}_3 z + \widetilde{a}_6 z^2, \qquad \widetilde{A}_{01}(z) = \widetilde{a}_1 + \widetilde{a}_4 z + \widetilde{a}_7 z^2, \qquad \widetilde{A}_{02}(z) = \widetilde{a}_2 + \widetilde{a}_5 z + \widetilde{a}_8 z^2.$$

Therefore, for all $z \in \mathbb{T}$, we have

$$\begin{aligned}\sum_{k=0}^{3} A_{0k}(z)\overline{\widetilde{A}_{0k}(z)} = \frac{1}{3} &+ (a_0\overline{\widetilde{a}}_3 + a_1\overline{\widetilde{a}}_4 + a_2\overline{\widetilde{a}}_5)\overline{z} + (\overline{\widetilde{a}}_0 a_3 + \overline{\widetilde{a}}_1 a_4 + \overline{\widetilde{a}}_2 a_5)z + (a_0\overline{\widetilde{a}}_6 + a_1\overline{\widetilde{a}}_7 + a_2\overline{\widetilde{a}}_8)\overline{z}^2 \\ &+ (\overline{\widetilde{a}}_0 a_6 + \overline{\widetilde{a}}_1 a_7 + \overline{\widetilde{a}}_2 a_8)z^2 + (a_3\overline{\widetilde{a}}_6 + a_4\overline{\widetilde{a}}_7 + a_5\overline{\widetilde{a}}_8)z\overline{z}^2 + (\overline{\widetilde{a}}_3 a_6 + \overline{\widetilde{a}}_4 a_7 + \overline{\widetilde{a}}_5 a_8)\overline{z}z^2.\end{aligned}$$

Thus, condition (2.11) holds if and only if

$$a + \alpha + (\overline{\widetilde{\alpha}} + d)\varepsilon_3 + (\overline{\widetilde{a}} + \widetilde{d})\varepsilon_3^2 = \overline{\widetilde{a}} + \overline{\widetilde{\alpha}} + (\alpha + \widetilde{d})\varepsilon_3 + (a + d)\varepsilon_3^2 = 0,$$

$$a + \alpha + (\overline{\widetilde{a}} + \widetilde{d})\varepsilon_3 + (\overline{\widetilde{\alpha}} + d)\varepsilon_3^2 = \overline{\widetilde{a}} + \overline{\widetilde{\alpha}} + (a + d)\varepsilon_3 + (\alpha + \widetilde{d})\varepsilon_3^2 = 0,$$

$$d + \widetilde{d} + (a + \overline{\widetilde{a}})\varepsilon_3 + (\alpha + \overline{\widetilde{\alpha}})\varepsilon_3^2 = 0,$$

where $d = a\overline{\widetilde{\alpha}} + b\overline{\widetilde{\beta}} + c\overline{\widetilde{\gamma}}$ and $\widetilde{d} = \overline{\widetilde{a}}\alpha + \overline{\widetilde{b}}\beta + \overline{\widetilde{c}}\gamma$ (cf. the algorithms for constructing wavelets in [13, Section 2.6; 18–20]).

Remark 1. Conditions for the functions $\psi^{(\nu)}$ and $\widetilde{\psi}^{(\nu)}$, $\nu = 1, \dots, p-1$, defined by (2.9) to generate frames or Riesz bases in $L^2(G)$ can be stated by analogy with Theorem 2.7.5 in [13]. For example, it suffices to assume that $\widehat{\rho}\,[m] = \widehat{\rho}\,[\widetilde{m}] = 0$; then, for the refinable functions φ and $\widetilde{\varphi}$, expansions of the form (1.8) contain only finitely many nonzero coefficients, and their Fourier transforms $\widehat{\varphi}$ and $\widehat{\widetilde{\varphi}}$ have compact support (see [10, Proposition 2]). In this case, under conditions (2.2), (2.8), and

$$m_\nu(\theta) = 0, \qquad \widetilde{m}_\nu(\theta) = 0, \qquad \nu = 1, \dots, p-1,$$

each of the systems $\{\psi^{(\nu)}_{j,h}\}$ and $\{\widetilde{\psi}^{(\nu)}_{j,h}\}$ is a frame in $L^2(G)$. If the systems $\{\varphi(\,\cdot\ominus h) \mid h \in H\}$ and $\{\widetilde{\varphi}(\,\cdot\ominus h) \mid h \in H\}$ are in addition biorthonormal, then the functions $\psi^{(\nu)}$ and $\widetilde{\psi}^{(\nu)}$, $\nu = 1, \ldots, p-1$, form a biorthogonal wavelet set with respect to the pair $\{V_j\}$, $\{\widetilde{V}_j\}$, and each of the systems $\{\psi^{(\nu)}_{j,h}\}$ and $\{\widetilde{\psi}^{(\nu)}_{j,h}\}$ is a Riesz basis in $L^2(G)$.

Remark 2. The graphic illustrations presented in Examples 1 and 2 are given for parameter values maximizing the peak signal-to-noise ratio under the compression of *Lena* and *Bridge* images according to the procedure described in [21] as applied to orthogonal dyadic wavelets.

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