Orthogonal p-wavelets on $\mathbf{R}_+$

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Abstract

In this paper we give a general construction of compactly supported orthogonal p-wavelets in $L^2(\mathbf{R}_+)$ arising from scaling filters with p^n many terms. For all integer $p \geq 2$ these wavelets are identified with certain lacunary Walsh series on $\mathbf{R}_+$. The case where $p = 2$ was studied by W.C. Lang mainly from the point of view of the wavelet analysis on the Cantor dyadic group (the dyadic or 2-series local field). Our approach is connected with the Walsh – Fourier transform and the elements of M-band wavelet theory.

Keywords: compactly supported p-wavelets, M-band wavelet transform, Walsh functions, Walsh – Fourier transform, p-series local field, signal processing.

AMS Subject Classification: 42A38, 42A55, 42C15, 42C40, 43A70.

1 Introduction

It is well-known that the scaling function φ for a system of p-wavelets with compact support on the real line $\mathbf{R}$ satisfies a refinement equation of the type

$$\varphi(x) = \sum_{k=0}^{N} c_k \varphi(px - k).$$

The case where $p = 2$ have been studied by many authors in great detail (see, e.g., [1], [2], and references therein). The wavelet theory for integer values of

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$p > 2$ is also developed and have some applications in the image processing context (see [3] for the bibliography and [4] for more recent results). In this paper we give a general construction of compactly supported orthogonal p-wavelets on the positive half-line.

Let p be a fixed natural number greater than 1. As usual, let $\mathbf{R}_+ = [0, +\infty)$ and $\mathbf{Z}_+ = \{0, 1, \ldots\}$. Denote by $[x]$ the integer part of x. For $x \in \mathbf{R}_+$ and any positive integer j we set

$$x_j = [p^j x] (\operatorname{mod} p), \qquad x_{-j} = [p^{1-j} x] (\operatorname{mod} p), \tag{1.1}$$

where $x_j, x_{-j} \in \{0, 1, \ldots, p-1\}$. It is clear that for each $x \in \mathbf{R}_+$ there exists $k = k(x)$ in $\mathbf{N}$ such that $x_{-j} = 0$ for all $j > k$.

Consider on $\mathbf{R}_+$ the addition defined as follows: if $z = x \oplus y$, then

$$z = \sum_{j<0} \zeta_j p^{-j-1} + \sum_{j>0} \zeta_j p^{-j}$$

with

$$\zeta_j = x_j + y_j \,(\operatorname{mod} p) \quad (j \in \mathbf{Z} \setminus \{0\}),$$

where $\zeta_j \in \{0, 1, \ldots, p-1\}$ and x_j, y_j are calculated by (1.1). We note that this binary operation appears in the study of the dyadic Hardy spaces on $\mathbf{R}_+$ (see, e.g., [5]). It also is implicit in the book [6] where $\oplus$ is used for the study of Walsh series and there applications in the image and date compression.

As usual, we write $z = x \ominus y$, if $z \oplus y = x$.

Let n be a positive integer. Consider a refinement equation of the type

$$\varphi(x) = p \sum_{\alpha=0}^{p^n-1} a_\alpha \varphi(px \ominus \alpha). \tag{1.2}$$

The case where $p = 2$ was studied by W.C. Lang [7]–[9] mainly from the point of view of the wavelet analysis on the Cantor dyadic group (the dyadic or 2-series local field). As noted in [9], the Cantor dyadic group is rather different in structure than other groups for which wavelet construction have been carried out.

Denote by $\mathbf{1}_E$ the characteristic function of a subset E of $\mathbf{R}_+$. It was shown in [7] that the function φ defined by

$$\varphi(x) = (1/2) \mathbf{1}_{[0,1)}(x/2) (1 + a \sum_{j=0}^{\infty} b^j w_{2^{j+1}-1}(x/2)), \quad x \in \mathbf{R}_+, \tag{1.3}$$

where $0 < a \le 1, a^2 + b^2 = 1$, and $\{w_m(x)\}$ is the classical Walsh system, satisfies the equation (1.2) with $p = n = 2$ and

$$a_0 = (1{+}a{+}b)/4, \quad a_1 = (1{+}a{-}b)/4, \quad a_2 = (1{-}a{-}b)/4, \quad a_3 = (1{-}a{+}b)/4.$$

Moreover, the corresponding wavelet is given by

$$\psi(x) = 2a_0\varphi(2x \ominus 1) - 2a_1\varphi(2x) + 2a_2\varphi(2x \ominus 3) - 2a_3\varphi(2x \ominus 2).$$

Our purpose here is to extend these results for all possible p and n. In particular, a method is given to construct a system of wavelets that can be used to decompose functions in $L^2(\mathbf{R}_+)$ which is based on decimation by an integer $p > 2$. Among motivations can be point out applications of the p-series local fields to digital signal processing (e.g., [6, Ch.12]) and the similar results in the wavelet theory on the line $\mathbf{R}$ (see, e.g., [10]). Our main results, Theorems 1 and 2, may be strengthened to provide $p-$wavelet bases for spaces beyond $L^2(\mathbf{R}_+)$ (cf. [7, § 4], [9, § 4]).

For integer $n \ge 2$, we denote by $\mathbf{N}_0(p, n)$ the set of all natural numbers $m \ge p^{n-1}$ for which in the p-ary expansion

$$m = \sum_{j=0}^{k} \mu_j p^j, \quad \mu_j \in \{0, 1, \dots, p-1\}, \quad \mu_k \ne 0, \quad k = k(m) \in \mathbf{Z}_+, \quad (1.4)$$

there is no n-tuple $(\mu_j, \mu_{j+1}, \dots, \mu_{j+n-1})$ that coincides with some of the n-tuples

$$(0, 0, \dots, 0, 1), (0, 0, \dots, 0, 2), \dots, (0, 0, \dots, 0, p-1).$$

Put $\mathbf{N}(p, n) = \{1, 2, \dots, p^{n-1} - 1\} \cup \mathbf{N}_0(p, n)$. For example:

$$\mathbf{N}(2, 2) = \{2^{j+1} - 1 \mid j \in \mathbf{Z}_+\} = \{1, 3, 7, 15, 31, \dots\},$$

$$\mathbf{N}(2, 3) = \{1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 21, \dots\},$$

$$\mathbf{N}(3, 2) = \{\sum_{j=0}^{k} m_j 3^j \mid m_j \in \{1, 2\}, k \in \mathbf{Z}_+\} = \{1, 2, 4, 5, 7, 8, 13, \dots\}.$$

For every $m \in \mathbf{N}(p, n), 1 \le m \le p^n - 1$, we choose a (real or complex) number b_m in such a way that

$$b_j \ne 0 \quad \text{and} \quad |b_j|^2 + |b_{p^{n-1}+j}|^2 + |b_{2p^{n-1}+j}|^2 + \dots + |b_{(p-1)p^{n-1}+j}|^2 = 1 \quad (1.5)$$

for $j = 1, 2, \ldots, p^{n-1} - 1$. In the case $p = n = 2$ we have only one equality: $|b_1|^2 + |b_3|^2 = 1$. Also, it is easy to see that

$$|b_1|^2 + |b_5|^2 = |b_2|^2 + |b_6|^2 = |b_3|^2 + |b_7|^2 = 1, \quad \text{if} \quad p = 2, n = 3$$

and

$$|b_1|^2 + |b_4|^2 + |b_7|^2 = |b_2|^2 + |b_5|^2 + |b_8|^2 = 1, \quad \text{if} \quad p = 3, n = 2.$$

The condition (1.5) is necessary for the orthonormality of our system $\{\varphi(\cdot \ominus k) \mid k \in \mathbf{Z}_+\}$ (see assertion (b) in Theorem 2 below).

For $m \in \mathbf{N}(p, n), 1 \le m \le p^n - 1$, we set

$$c(i_1, i_2, \ldots, i_n) = b_m, \quad \text{if} \quad m = i_1 p^0 + i_2 p^1 + \ldots + i_n p^{n-1},\ i_j \in \{0, 1, \ldots, p-1\}.$$

Then for $m \in \mathbf{N}(p, n)$ using p-ary expansion (1.4) we write:

$$A(m) = c(\mu_0, 0, 0, \ldots, 0, 0), \quad \text{if} \quad k(m) = 0;$$

$$A(m) = c(\mu_1, 0, 0, \ldots, 0, 0) c(\mu_0, \mu_1, 0, \ldots, 0, 0), \quad \text{if} \quad k(m) = 1;$$

$$\ldots$$

$$A(m) = c(\mu_k, 0, 0, \ldots, 0, 0) c(\mu_{k-1}, \mu_k, 0, \ldots, 0, 0) \ldots$$
$$\ldots c(\mu_0, \mu_1, \mu_2, \ldots, \mu_{n-2}, \mu_{n-1}), \quad \text{if} \quad k = k(m) \ge n - 1.$$

And for $s \in \{0, 1, \ldots, p^n - 1\}$ we put

$$d_s^{(n)} = \begin{cases} 1, & \text{if} \quad s = 0, \\ b_s, & \text{if} \quad s = j + l p^{n-1}\ (1 \le j \le p^{n-1} - 1,\ 0 \le l \le p - 1), \\ 0, & \text{if} \quad s = p^n - l p^{n-1}\ (1 \le l \le p - 1). \end{cases}$$

For $x \in [0, 1)$, let $r_0(x)$ be given by

$$r_0(x) = \begin{cases} 1, & \text{if} \quad x \in [0, 1/p), \\ \varepsilon_p^l, & \text{if} \quad x \in [l p^{-1}, (l+1) p^{-1})\ (l = 1, \ldots, p - 1), \end{cases}$$

where $\varepsilon_p = \exp(2\pi i/p)$. The extension of the function r_0 to $\mathbf{R}_+$ is defined by the equality $r_0(x+1) = r_0(x), x \in \mathbf{R}_+$. Then *the generalized Walsh functions* $\{w_m(x)\}(m \in \mathbf{Z}_+)$ are defined by

$$w_0(x) \equiv 1, \qquad w_m(x) = \prod_{j=0}^{k} (r_0(p^j x))^{\mu_j},$$

where

$$m = \sum_{j=0}^{k} \mu_j p^j, \quad \mu_j \in \{0, 1, \dots, p-1\}, \quad \mu_k \neq 0$$

(the classical Walsh system corresponds to the case $p = 2$).

For $x, \omega \in \mathbf{R}_+$, let

$$\chi(x, \omega) = \exp\Big(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j \omega_{-j} + x_{-j}\omega_j)\Big), \tag{1.6}$$

where x_j, ω_j are given by (1.1). Note that $\chi(x, m/p^{n-1}) = \chi(x/p^{n-1}, m) = w_m(x/p^{n-1})$ for all $x \in [0, p^{n-1})$, $m \in \mathbf{Z}_+$.

Theorem 1. *The function φ given by the formula*

$$\varphi(x) = (1/p^{n-1})\mathbf{1}_{[0,1)}(x/p^{n-1})(1 + \sum_{m \in \mathbf{N}(p,n)} A(m) w_m(x/p^{n-1})), \quad x \in \mathbf{R}_+, \tag{1.7}$$

is a solution of the refinement equation (1.2) *provided* $\{a_\alpha\}$ *satisfy the linear equations*

$$\sum_{\alpha=0}^{p^n-1} a_\alpha \overline{\chi(\alpha, sp^{-n})} = d_s^{(n)} \quad (0 \leq s \leq p^n - 1). \tag{1.8}$$

Moreover, the system $\{\varphi(\cdot \ominus k) \mid k \in \mathbf{Z}_+\}$ is orthonormal in $L^2(\mathbf{R}_+)$.

It is easily seen that (1.7) coincides with (1.3) when $p = n = 2$ and $b_1 = a, b_3 = b$.

Example 1. Suppose that $b_1 = b_2 = \ldots = b_{p^{n-1}-1} = 1$. Then, by (1.5), $b_m = 0$ for $m \geq p^{n-1}$ and hence

$$A(m) = \begin{cases} 1, & \text{if} \quad m \in \{1, \dots, p^{n-1} - 1\}, \\ 0, & \text{if} \quad m \in \mathbf{N}_0(p, n). \end{cases}$$

Since

$$\sum_{m=0}^{p^{n-1}-1} \chi(y,m) = \begin{cases} p^{n-1}, & \text{if} \quad 0 \le y < 1/p^{n-1}, \\ 0, & \text{if} \quad 1/p^{n-1} \le y < 1 \end{cases}$$

(see [6, § 1.5]), we have $\varphi = \mathbf{1}_{[0,p^{n-1})}$. This function satisfies the equation (1.2) when $a_0 = \ldots = a_{p-1} = 1/p$ and $a_\alpha = 0$ for $\alpha \ge p$. Note that Theorem 1 is still true for $n = 1$ (the Haar case), if we assume $\mathbf{N}(p,1) = \emptyset$.

Example 2 (cf. [9, § 5.4]). Suppose φ is given by (1.7) with $p = 2, n = 3$, and

$$b_1 = a,\ b_2 = b,\ b_3 = c,\ b_5 = \alpha,\ b_6 = \beta,\ b_7 = \gamma,$$

where

$$|a|^2 + |\alpha|^2 = |b|^2 + |\beta|^2 = |c|^2 + |\gamma|^2 = 1.$$

Then φ satisfies the equation

$$\varphi(x) = 2\sum_{j=0}^{7} a_j \varphi(2x \ominus j)$$

with the coefficients

$$a_0 = \frac{1}{8}(1 + a + b + c + \alpha + \beta + \gamma),$$

$$a_1 = \frac{1}{8}(1 + a + b + c - \alpha - \beta - \gamma),$$

$$a_2 = \frac{1}{8}(1 + a - b - c + \alpha - \beta - \gamma),$$

$$a_3 = \frac{1}{8}(1 + a - b - c - \alpha + \beta + \gamma),$$

$$a_4 = \frac{1}{8}(1 - a + b - c - \alpha + \beta - \gamma),$$

$$a_5 = \frac{1}{8}(1 - a + b - c + \alpha - \beta + \gamma),$$

$$a_6 = \frac{1}{8}(1 - a - b + c - \alpha - \beta + \gamma),$$

$$a_7 = \frac{1}{8}(1 - a - b + c + \alpha + \beta - \gamma).$$

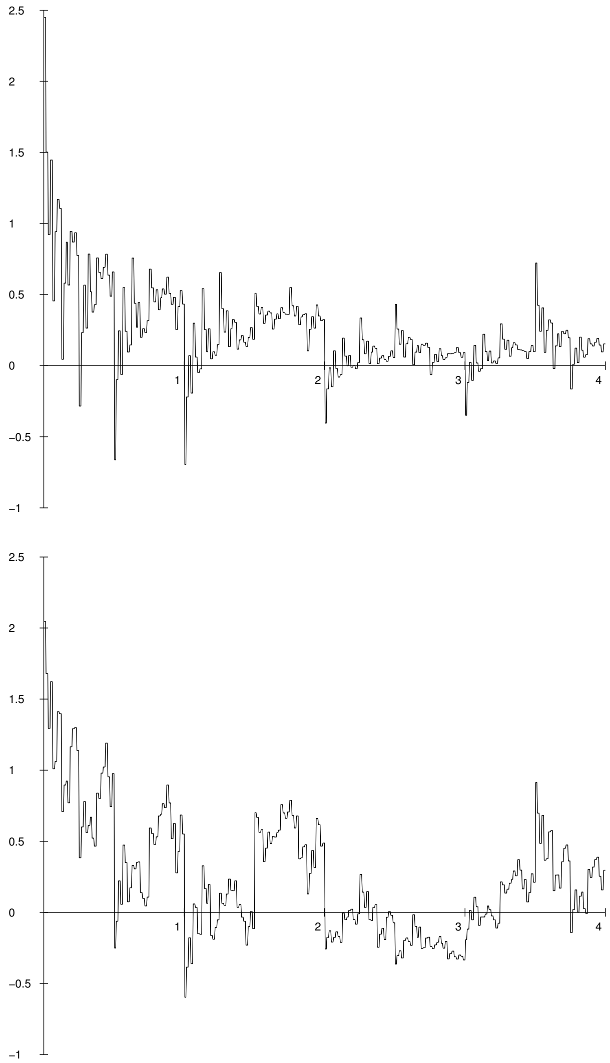

Figure 1: The scaling functions of example 2 for $a = 0.6$, $b = 0.4$, $c = 0.57$, $\alpha = 0.8$, $\beta = 0.9165$, $\gamma = 0.8216$ (top) and for $a = 0.9$, $b = 0.1$, $c = 0.87$, $\alpha = 0.4359$, $\beta = 0.9499$, $\gamma = 0.4931$ (bottom).

We give graphs of φ for certain values of $a, b, c, \alpha, \beta, \gamma$ (see Figs. 1– 2). All plottings were generated using MatLab 6.5.

Example 3. Let φ be given by (1.7) with $p = 3, n = 2$, and

$$b_1 = a,\ b_2 = \alpha,\ b_4 = b,\ b_5 = \beta,\ b_7 = c,\ b_8 = \gamma,$$

where

$$|a|^2 + |b|^2 + |c|^2 = |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1.$$

Then φ satisfies the equation

$$\varphi(x) = 3\sum_{j=0}^{8} a_j \varphi(3x \ominus j)$$

with the coefficients

$$a_0 = \frac{1}{9}(1 + a + b + c + \alpha + \beta + \gamma),$$

$$a_1 = \frac{1}{9}(1 + a + \alpha + (b + \beta)\varepsilon_3^2 + (c + \gamma)\varepsilon_3),$$

$$a_2 = \frac{1}{9}(1 + a + \alpha + (b + \beta)\varepsilon_3 + (c + \gamma)\varepsilon_3^2),$$

$$a_3 = \frac{1}{9}(1 + (a + b + c)\varepsilon_3^2 + (\alpha + \beta + \gamma)\varepsilon_3),$$

$$a_4 = \frac{1}{9}(1 + c + \beta + (a + \gamma)\varepsilon_3^2 + (b + \alpha)\varepsilon_3),$$

$$a_5 = \frac{1}{9}(1 + b + \gamma + (a + \beta)\varepsilon_3^2 + (c + \alpha)\varepsilon_3),$$

$$a_6 = \frac{1}{9}(1 + (a + b + c)\varepsilon_3 + (\alpha + \beta + \gamma)\varepsilon_3^2),$$

$$a_7 = \frac{1}{9}(1 + b + \gamma + (a + \beta)\varepsilon_3 + (c + \alpha)\varepsilon_3^2),$$

$$a_8 = \frac{1}{9}(1 + c + \beta + (a + \gamma)\varepsilon_3 + (b + \alpha)\varepsilon_3^2),$$

where $\varepsilon_3 = \exp(2\pi i/3)$.

We note, that for all p, n

$$a_\alpha = \frac{1}{p^n}\sum_{s=0}^{p^n-1} d_s^{(n)} \chi(\alpha, sp^{-n}) \quad (0 \le \alpha \le p^n - 1), \tag{1.9}$$

which follows from (1.8). This relation is an analogue of the inverse discrete Fourier transform (for the corresponding fast algorithm see, e.g., [11, p.459]).

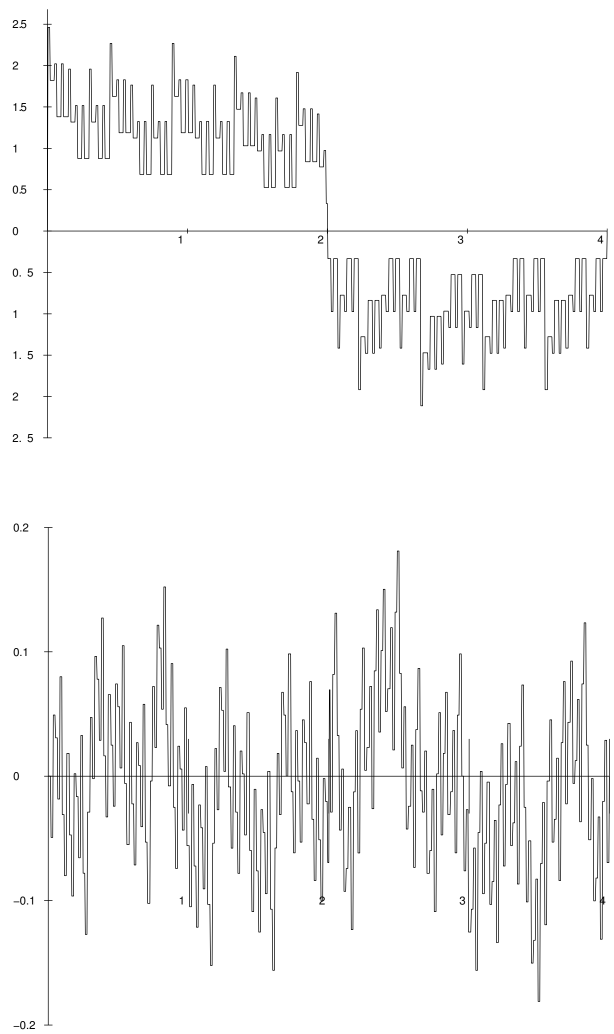

Figure 2: The real part (top) and imaginary part (bottom) of scaling function φ from example 3 with $a = 0.3$, $b = 0.5$, $c = 0.8124$, $\alpha = 0.4$, $\beta = 0.7$, $\gamma = 0.5916$.

2 The Walsh – Fourier transform and multiresolution p-analysis

The Walsh – Fourier transform of a function $f \in L^1(\mathbf{R}_+)$ is defined by

$$\tilde{f}(\omega) = \int_{\mathbf{R}_+} f(x)\overline{\chi(x,\omega)}dx,$$

where $\chi(x,\omega)$ is given by (1.6). If $f \in L^2(\mathbf{R}_+)$ and

$$J_a f(\omega) = \int_0^a f(x)\overline{\chi(x,\omega)}dx \quad (a > 0),$$

then $\tilde{f}$ is defined as the limit of $J_a f$ in $L^2(\mathbf{R}_+)$ as $a \to \infty$.

The properties of the Walsh – Fourier transform are quite similar to those of the classical Fourier transform (see, e.g., [6, Ch.6] or [11, Ch.9]). In particular, if $f \in L^2(\mathbf{R}_+)$, then $\tilde{f} \in L^2(\mathbf{R}_+)$ and

$$||\tilde{f}||_{L^2(\mathbf{R}_+)} = ||f||_{L^2(\mathbf{R}_+)}.$$

If $x, y, \omega \in \mathbf{R}_+$ and $x \oplus y$ is p-adic irrational, than

$$\chi(x \oplus y, \omega) = \chi(x,\omega)\chi(y,\omega), \tag{2.1}$$

(see [6, § 1.5]). Thus, for fixed x and ω, the equality (2.1) holds for all $y \in \mathbf{R}_+$ except for countably many. It is known also, that the systems $\{\chi(\alpha,\cdot)\}_{\alpha=0}^{\infty}$ and $\{\chi(\cdot,\alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2[0,1]$.

Accoding to [6, § 6.2] for any $\varphi \in L^2(\mathbf{R}_+)$ we have

$$\int_{\mathbf{R}_+} \varphi(x)\overline{\varphi(x \ominus k)}dx = \int_{\mathbf{R}_+} |\tilde{\varphi}(\omega)|^2\overline{\chi(k,\omega)}d\omega, \quad k \in \mathbf{Z}_+. \tag{2.2}$$

Let us denote by $\{\omega\}$ the fractional part of ω. For $k \in \mathbf{Z}_+$, we have $\chi(k,\omega) = \chi(k,\{\omega\})$. Thus from (2.2) it follows that

$$\int_{\mathbf{R}_+} \varphi(x)\overline{\varphi(x \ominus k)}dx = \sum_{l=0}^{\infty}\int_l^{l+1} |\tilde{\varphi}(\omega)|^2\overline{\chi(k,\{\omega\})}d\omega$$

$$= \int_0^1 \left(\sum\nolimits_{l\in\mathbf{Z}_+} |\tilde{\varphi}(\omega + l)|^2\right)\overline{\chi(k,\omega)}d\omega.$$

Therefore, a necessary and sufficient condition for a system $\{\varphi(\cdot \ominus k) \mid k \in \mathbf{Z}_+\}$ to be orthonormal in $L^2(\mathbf{R}_+)$ is

$$\sum\nolimits_{l\in\mathbf{Z}_+} |\tilde{\varphi}(\omega + l)|^2 = 1 \quad \text{a.e.} \tag{2.3}$$

Definition. *A multiresolution p-analysis in* $L^2(\mathbf{R}_+)$ is a sequence of closed subspaces $V_j \subset L^2(\mathbf{R}_+)(j \in \mathbf{Z})$ such that the following hold:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbf{Z}$.

(ii) The union $\bigcup V_j$ is dense in $L^2(\mathbf{R}_+)$, and $\bigcap V_j = \{0\}$.

(iii) $f(\cdot) \in V_j \iff f(p\,\cdot) \in V_{j+1}$ for all $j \in \mathbf{Z}$.

(iv) $f(\cdot) \in V_0 \implies f(\cdot \oplus k) \in V_0$ for all $k \in \mathbf{Z}_+$.

(v) There is a function $\varphi \in L^2(\mathbf{R}_+)$ such that $\{\varphi(\cdot \ominus k) \mid k \in \mathbf{Z}_+\}$ is an orthonormal basis of V_0.

The function φ is called *a scaling function* in $L^2(\mathbf{R}_+)$.

By conditions (v) and (iii) the functions $\varphi_{1,k}(x) = p^{1/2}\varphi(px \ominus k) \quad (k \in \mathbf{Z}_+)$ constitude an orthonormal basis in V_1. Since $V_0 \subset V_1$, the scaling function φ belongs to V_1 and has the Fourier expansion

$$\varphi(x) = \sum_{k\in\mathbf{Z}_+} h_k p^{1/2}\varphi(px \ominus k), \quad h_k = \int\limits_{\mathbf{R}_+} \varphi(x)\overline{\varphi_{1,k}(x)}dx.$$

This implies that

$$\varphi(x) = p\sum_{\alpha\in\mathbf{Z}_+} a_\alpha \varphi(px \ominus \alpha), \quad \sum_{\alpha\in\mathbf{Z}_+} |a_\alpha|^2 < +\infty, \tag{2.4}$$

where $a_\alpha = p^{-1/2}h_\alpha$. Under the Walsh – Fourier transform we have

$$\tilde{\varphi}(\omega) = m_0(p^{-1}\omega)\tilde{\varphi}(p^{-1}\omega), \tag{2.5}$$

where

$$m_0(\omega) = \sum_{\alpha\in\mathbf{Z}_+} a_\alpha \overline{\chi(\alpha, \omega)}. \tag{2.6}$$

When $p = 2$ a function ψ given by the formula

$$\psi(x) = 2\sum_{\alpha\in\mathbf{Z}_+} (-1)^\alpha \overline{a}_\alpha \varphi(2x \ominus (\alpha \oplus 1)) \tag{2.7}$$

is a wavelet in $L^2(\mathbf{R}_+)$, associated with the scaling function φ. Therefore, the system of functions

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x \ominus k) \quad (j \in \mathbf{Z},\, k \in \mathbf{Z}_+)$$

is an orthonormal bases in $L^2(\mathbf{R}_+)$ (cf. [1, § 5.1] and [9, § 3]).

Let $p > 2$. It follows from (2.3) and (2.5) that

$$|m_0(\omega)|^2 + |m_0(\omega + 1/p)|^2 + \ldots + |m_0(\omega + (p-1)/p)|^2 = 1 \tag{2.8}$$

for a.e. $\omega \in [0, 1)$. Suppose that we have $p - 1$ functions

$$m_l(\omega) = \sum_{\alpha \in \mathbf{Z}_+} a_\alpha^{(l)} \overline{\chi(\alpha, \omega)}, \qquad \sum_{\alpha \in \mathbf{Z}_+} |a_\alpha^{(l)}|^2 < +\infty \quad (1 \le l \le p-1),$$

such that

$$(m_l(\omega + k/p))_{l,k=0}^{p-1} \tag{2.9}$$

is a unitary matrix for a.e. $\omega \in [0, 1)$ (for the problem of unitary extension see, e.g., [10],[12],[13]). We set

$$\psi_l(x) = p \sum_{\alpha \in \mathbf{Z}_+} a_\alpha^{(l)} \varphi(px \ominus \alpha) \tag{2.10}$$

and

$$W_0^{(l)} = \mathrm{clos}_{L^2(\mathbf{R}_+)} \mathrm{span}\, \{\psi_l(\cdot \ominus k) \mid k \in \mathbf{Z}_+\} \quad (1 \le l \le p-1).$$

Let W_j be the orthogonal complement of V_j in V_{j+1}. Then

$$W_0 = \bigoplus_{l=1}^{p-1} W_0^{(l)}, \quad L^2(\mathbf{R}_+) = \bigoplus_{j \in \mathbf{Z}} W_j,$$

where $\bigoplus$ denotes the orthogonal direct sum with the inner product of $L^2(\mathbf{R}_+)$. Moreover, the system of functions

$$\psi_{j,k,l}(x) = p^{j/2}\psi_l(p^j x \ominus k) \quad (j \in \mathbf{Z},\, k \in \mathbf{Z}_+, 1 \le l \le p-1) \tag{2.11}$$

is an orthonormal bases in $L^2(\mathbf{R}_+)$ (cf. [10], [14]).

3 Construction of p-wavelets

Let $\varphi \in L^2(\mathbf{R}_+)$ satisfies the refinement equation (1.2). As before, we get

$$\tilde{\varphi}(\omega) = m_0(p^{-1}\omega)\tilde{\varphi}(p^{-1}\omega), \tag{3.1}$$

where

$$m_0(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{\chi(\alpha,\omega)}. \tag{3.2}$$

Suppose that

$$m_0(0) = 1 \qquad (\text{i.e.} \quad \sum_{\alpha=0}^{p^n-1} a_\alpha = 1).$$

Put

$$\Delta_s^{(n)} := [sp^{-n}, (s+1)p^{-n}) \quad \text{for} \quad s \in \mathbf{Z}_+.$$

Then $m_0(\omega)$ is a constant on $\Delta_s^{(n)}$ for each s and $m_0(\omega) = 1$ on $\Delta_0^{(n)}$. It follows from (3.1) that

$$\tilde{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(p^{-j}\omega), \quad \omega \in \mathbf{R}_+. \tag{3.3}$$

We note that $m_0(p^{-j}\omega) = 1$ as $p^{-j}\omega \in \Delta_0^{(n)}$ (so product (3.3) is finite for every $\omega \in \mathbf{R}_+$).

We say that a function $f : \mathbf{R}_+ \mapsto \mathbf{C}$ is *W-continuous at a point* $x \in \mathbf{R}_+$, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x \oplus y) - f(x)| < \varepsilon$ for $0 < y < \delta$. It is known that $\chi(\alpha,\cdot)(\alpha \in \mathbf{Z}_+)$ are W-continuous functions. By (3.2) and (3.3), the same is true for m_0 and $\tilde{\varphi}$. Moreover, m_0 and $\tilde{\varphi}$ are uniformly W-continuous in [0,1) (cf. [6, § 2.3], [11, § 9.2]).

The collection $\{[0, p^{-j}) \,|\, j \in \mathbf{Z}\}$ is a fundamental system of neighborhoods of zero in the W-topology on $\mathbf{R}_+$ (cf. [6, § 1.2]).

Suppose that E is a W-compact set in $\mathbf{R}_+$. The notation $E \equiv [0,1)$ (mod $\mathbf{Z}_+$) means that for each $x \in [0,1)$ there exists $k \in \mathbf{Z}_+$ such that $x \oplus k \in E$. Denote by μ the Lebesgue measure on $\mathbf{R}_+$.

We can now state the analogue of Cohen's theorem (cf. [1, § 6.3] and [2, § 9.5]):

Theorem 2. *Let*

$$m_0(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{\chi(\alpha,\omega)}$$

be a polynomial satisfying the following conditions:

(a) $m_0(0) = 1$.

(b) $\sum_{j=0}^{p-1} |m_0(sp^{-n} \oplus jp^{-1})|^2 = 1$ *for* $s = 0, 1, \ldots, p^{n-1} - 1$.

(c) *There exists a* W*-compact set* E *such that* $\mathrm{int}\,(E) \ni 0, \mu(E) = 1, E \equiv [0,1)(\mathrm{mod}\ \mathbf{Z}_+)$, *and*

$$\inf_{j\in\mathbf{N}} \inf_{\omega\in E} |m_0(p^{-j}\omega)| > 0. \tag{3.4}$$

If the Walsh – Fourier transform of $\varphi \in L^2(\mathbf{R}_+)$ *can be written as*

$$\tilde{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(p^{-j}\omega), \tag{3.5}$$

then φ *is a scaling function in* $L^2(\mathbf{R}_+)$.

Remark 1. Assertion (b) of Theorem 2 is nothing but the statement that for our polynomial m_0 the equality (2.8) is true.

Remark 2. It is easy to check that m_0 with the coefficients $\{a_\alpha\}$ from (1.8) satisfies all conditions of Theorem 2. For example, since

$$m_0(\omega) \neq 0 \quad \text{for all} \quad \omega \in [0, 1/p),$$

condition (c) holds for $E = [0, 1)$. Therefore, the function φ given by (1.7) is a scaling function in $L^2(\mathbf{R}_+)$.

Theorems 1 and 2 tell us a general procedure to design p-wavelets in $L^2(\mathbf{R}_+)$:

1. Choose a set of numbers $\{b_m : m \in \mathbf{N}(p, n), 1 \leq m \leq p^n - 1\}$ so that (1.5) is true.
2. Compute $\{a_\alpha\}$ by (1.9).
3. With m_0 defined by (3.2) find

$$m_l(\omega) = \sum_{\alpha\in\mathbf{Z}_+} a_\alpha^{(l)} \overline{\chi(\alpha, \omega)}, \quad (1 \leq l \leq p - 1),$$

such that $(m_l(\omega + k/p))_{l,k=0}^{p-1}$ is a unitary matrix.

4. Define $\psi_1, \ldots, \psi_{p-1}$ by (2.10).

Remark 3. For wavelet construction the condition $b_j \neq 0$ in (1.5) can be replaced by assertion (c) of Theorem 2.

In connection with Remark 3 we give the following

Example 4. Let $p = 2$, $n = 3$ and

$$m_0(\omega) = \begin{cases} 1, & \text{if} \quad x \in [0, 1/4) \cup [3/8, 1/2) \cup [3/4, 7/8), \\ 0, & \text{if} \quad x \in [1/4, 3/8) \cup [1/2, 3/4) \cup [7/8, 1). \end{cases}$$

Then from (3.3) we see that $\tilde{\varphi} = \mathbf{1}_E$ where $E = [0, 1/2) \cup [3/4, 1) \cup [3/2, 7/4)$. Under the inverse Walsh – Fourier transform we obtain

$$\varphi(x) = \frac{1}{2}\mathbf{1}_{[0,2)}(x) + \frac{1}{4}\mathbf{1}_{[0,4)}(x)[w_3(x/4) + w_6(x/4)]$$

(see also Example 2 for $a = c = 1$, $b = 0$). By Theorem 2, this function generate a multiresolution 2-analysis in $L^2(\mathbf{R}_+)$.

4 Proofs

To prove Theorems 1 and 2, we need the following lemma (cf. [1, § 6.3] and [2, § 9.5]):

Lemma 1. *Under the conditions of Theorem 2 the system* $\{\varphi(\cdot \ominus k) \mid k \in \mathbf{Z}_+\}$ *is orthonormal in* $L^2(\mathbf{R}_+)$.

Proof. For $l \in \mathbf{N}$ let

$$\mu^{[l]}(\omega) = \prod_{j=1}^{l} m_0(\omega/p^j)\mathbf{1}_E(\omega/p^l), \quad \omega \in \mathbf{R}_+.$$

Since $0 \in \operatorname{int}(E)$ and $m_0(\omega) = 1$ on $\Delta_0^{(n)}$, we obtain from (3.5)

$$\lim_{l\to\infty} \mu^{[l]}(\omega) = \tilde{\varphi}(\omega), \quad \omega \in \mathbf{R}_+. \tag{4.1}$$

Also, by (a) and (c), there exists a number j_0 such that

$$m_0(\omega/p^j) = 1 \quad \text{for} \quad j > j_0,\, \omega \in E.$$

Thus,

$$\tilde{\varphi}(\omega) = \prod_{j=1}^{j_0} m_0(p^{-j}\omega), \quad \omega \in E.$$

By (3.4), there is a constant $c_1 > 0$ such that

$$|m_0(\omega/p^j)| \geq c_1 \quad \text{for} \quad j \in \mathbf{N},\, \omega \in E,$$

and so

$$c_1^{-j_0}|\tilde{\varphi}(\omega)| \geq \mathbf{1}_E(\omega), \quad \omega \in \mathbf{R}_+.$$

Therefore

$$|\mu^{[l]}(\omega)| = \prod_{j=1}^{l} |m_0(\omega/p^j)| \mathbf{1}_E(\omega/p^l) \leq c_1^{-j_0} \prod_{j=1}^{l} |m_0(\omega/p^j)||\tilde{\varphi}(\omega/p^l)|$$

which by (3.5) yields

$$|\mu^{[l]}(\omega)| \leq c_1^{-j_0}|\tilde{\varphi}(\omega)| \quad \text{for} \quad l \in \mathbf{N},\, \omega \in \mathbf{R}_+. \tag{4.2}$$

Now, for $l \in \mathbf{N}$ we define

$$I_l(s) := \int_{\mathbf{R}_+} |\mu^{[l]}(\omega)|^2 \overline{\chi(s,\omega)} d\omega, \quad s \in \mathbf{Z}_+.$$

Setting $E_l := \{\omega \in \mathbf{R}_+ |\, p^{-l}\omega \in E\}$ and $\zeta = p^{-l}\omega$, we have

$$\begin{aligned} I_l(s) = \textstyle\int_{E_l} \prod\limits_{j=1}^{l} |m_0(\omega/p^j)|^2 \overline{\chi(s,\omega)} d\omega = \\ = p^l \textstyle\int_E |m_0(\zeta)|^2 \prod\limits_{j=1}^{l-1} |m_0(p^j\zeta)|^2 \overline{\chi(s, p^l\zeta)} d\zeta, \end{aligned} \tag{4.3}$$

where the last integrand is 1-periodic.

Using the assumption $E \equiv [0,1)(\text{mod } \mathbf{Z}_+)$ and changing the variable we get from (4.3)

$$I_l(s) = p^{l-1} \int_0^1 \sum_0^{p-1} |m_0(\xi/p + i/p)|^2 \prod_{j=1}^{l-1} |m_0(p^{j-1}\xi)|^2 \overline{\chi(s, p^{l-1}\xi)} d\xi.$$

Therefore, in view of Remark 1,

$$I_l(s) = p^{l-1} \int_0^1 \prod_{j=0}^{l-2} |m_0(p^j\xi)|^2 \overline{\chi(s, p^{l-1}\xi)} d\xi$$

which by (4.3) becomes

$$I_l(s) = I_{l-1}(s).$$

Since

$$I_1(s) = p\int_0^1 |m_0(\xi)|^2\overline{\chi(s,p\xi)}d\xi = \int_0^1 \overline{\chi(s,\xi)}d\xi = \delta_{0,s},$$

where $\delta_{0,s}$ is the Kronecker delta, we get

$$I_l(s) = \delta_{0,s} \quad (l \in \mathbf{N}, s \in \mathbf{Z}_+). \tag{4.4}$$

In particular, for all $l \in \mathbf{N}$

$$I_l(0) = \int_{\mathbf{R}_+} |\mu^{[l]}(\omega)|^2 d\omega = 1.$$

By (4.1) and Fatou's lemma we then obtain

$$\int_{\mathbf{R}_+} |\tilde{\varphi}(\omega)|^2 d\omega \le 1.$$

Thus, from (4.1) and (4.2) by Lebesque's dominated convergence theorem it follows that

$$\int_{\mathbf{R}_+} |\tilde{\varphi}(\omega)|^2\overline{\chi(s,\omega)}d\omega = \lim_{l\to\infty} I_l(s).$$

Hence by (2.2) and (4.4),

$$\int_{\mathbf{R}_+} \varphi(x)\overline{\varphi(x\ominus s)}dx = \delta_{0,s}, \quad s \in \mathbf{Z}_+.$$

Proof of Theorem 1. Put $X_{1-n} = \mathbf{1}_{[0,1/p^{n-1})}$. For any $x \in \mathbf{R}_+$ we have

$$\int_{\mathbf{R}_+} X_{1-n}(\omega\ominus m/p^{n-1})\chi(x,\omega)d\omega = \chi(x,m/p^{n-1})\int_0^{1/p^{n-1}} \chi(x,\omega)d\omega$$

$$= (1/p^{n-1})\mathbf{1}_{[0,1)}(x/p^{n-1})\chi(x,m/p^{n-1}) = (1/p^{n-1})\mathbf{1}_{[0,1)}(x/p^{n-1})w_m(x/p^{n-1}).$$

Consequently, taking the Walsh – Fourier transform of both sides of (1.7) gives

$$\tilde{\varphi}(\omega) = X_{1-n}(\omega) + \sum_{m\in\mathbf{N}(p,n)} A(m)X_{1-n}(\omega\ominus m/p^{n-1}). \tag{4.5}$$

If $\zeta \in [0, 1/p^{n-1})$ and $m \in \mathbf{N}(p, n)$, then clearly

$$\zeta \oplus \frac{m}{p^{n-1}} = \zeta + \frac{m}{p^{n-1}}$$

since $[p^{n+j}\zeta] (\mathrm{mod}\, p) = 0$ for all negative intergers j. Hence setting $\zeta = \omega \ominus m/p^{n-1}$, we see from (4.5) that

$$\tilde{\varphi}(\omega) = \begin{cases} 1, & \text{if} \quad \omega \in \Delta_0^{(n-1)}, \\ A(m), & \text{if} \quad \omega \in \Delta_m^{(n-1)}, \\ 0 & \text{otherwise}, \end{cases} \tag{4.6}$$

where $m \in \mathbf{N}(p, n)$.

Now, let the polynomial

$$m_0(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{\chi(\alpha, \omega)}$$

satisfy the condition (1.8), that is, $m_0(sp^{-n}) = d_s^{(n)}$ for $0 \le s \le p^n - 1$. Then, by the definition of $\{A(m)\}$, from (4.6) we have

$$\tilde{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(p^{-j}\omega)$$

and so

$$\tilde{\varphi}(\omega) = m_0(p^{-1}\omega)\tilde{\varphi}(p^{-1}\omega)$$

which gives (1.2). By Lemma 1 and Remark 2 the system $\{\varphi(\cdot \ominus k) \mid k \in \mathbf{Z}_+\}$ is orthonormal in $L^2(\mathbf{R}_+)$.

For integer m let $\mathcal{E}_m(\mathbf{R}_+)$ denotes the collection of all functions f on $\mathbf{R}_+$ which are constant on $[sp^{-m}, (s+1)p^{-m})$ for each $s \in \mathbf{Z}_+$. Further, we set

$$\tilde{\mathcal{E}}_m(\mathbf{R}_+) := \{f \ : \ f \text{ is } W\text{-continuous and } \tilde{f} \in \mathcal{E}_m(\mathbf{R}_+)\}$$

and

$$\mathcal{E}(\mathbf{R}_+) := \bigcup_{m=1}^{\infty} \mathcal{E}_m(\mathbf{R}_+), \quad \tilde{\mathcal{E}}(\mathbf{R}_+) := \bigcup_{m=1}^{\infty} \tilde{\mathcal{E}}_m(\mathbf{R}_+).$$

The following properties are true (see [6, § 6.2 and § 10.5]):

1. $\mathcal{E}(\mathbf{R}_+)$ and $\tilde{\mathcal{E}}(\mathbf{R}_+)$ are dense in $L^q(\mathbf{R}_+)$ for $1 \le q \le \infty$.
2. If $f \in L^1(\mathbf{R}_+) \cap \mathcal{E}_m(\mathbf{R}_+)$, then $\operatorname{supp} \tilde{f} \subset [0, p^m]$.
3. If $f \in L^1(\mathbf{R}_+) \cap \tilde{\mathcal{E}}_m(\mathbf{R}_+)$, then $\operatorname{supp} f \subset [0, p^m]$.

For $\varphi \in L^2(\mathbf{R}_+)$ we put

$$\varphi_{j,k}(x) = p^{j/2}\varphi(p^j x \ominus k) \quad (j \in \mathbf{Z},\, k \in \mathbf{Z}_+)$$

and

$$V_j = \operatorname{clos}_{L^2(\mathbf{R}_+)} \operatorname{span} \{\varphi_{j,k} |\ k \in \mathbf{Z}_+\} \quad (j \in \mathbf{Z}). \tag{4.7}$$

Let P_j be the orthogonal projection of $L^2(\mathbf{R}_+)$ to V_j. Also, we denote the norm in $L^2(\mathbf{R}_+)$ briefly by $||\cdot||$.

As an analogue of Proposition 5.3.1 in [1] (cf. Theorem 2.2 in [12]), we have:

Lemma 2. *If $\{\varphi_{0,k}\}$ is an orthogonal basis in V_0, then $\bigcap V_j = \{0\}$.*

Proof. Let $f \in \bigcap V_j$. Given an $\varepsilon > 0$ we choose $u \in L^1(\mathbf{R}_+) \cap \tilde{\mathcal{E}}(\mathbf{R}_+)$ such that $||f - u|| < \varepsilon$. Then

$$||f - P_j u|| \le ||P_j(f - u)|| \le ||f - u|| < \varepsilon$$

and so

$$||f|| \le ||P_j u|| + \varepsilon \tag{4.8}$$

for every $j \in \mathbf{Z}$.

Now, choose $R > 0$ so that $\operatorname{supp} u \subset [0, R)$. Then

$$(P_j u, \varphi_{j,k}) = (u, \varphi_{j,k}) = p^{j/2} \int_0^R u(x)\overline{\varphi(p^j x \ominus k)}dx.$$

Hence, by the Cauchy – Schwarz inequality,

$$||P_j u||^2 = \sum_{k \in \mathbf{Z}_+} |(P_j u, \varphi_{j,k})|^2 \le ||u||^2 \sum_{k \in \mathbf{Z}_+} p^j \int_0^R |\varphi(p^j x \ominus k)|^2 dx.$$

Therefore, if j is chosen small enough so that $Rp^j < 1$, then

$$||P_j u||^2 \le ||u||^2 \int_{S_{R,j}} |\varphi(x)|^2 dx = ||u||^2 \int_{\mathbf{R}_+} \mathbf{1}_{S_{R,j}}(x)|\varphi(x)|^2 dx, \tag{4.9}$$

where $S_{R,j} := \bigcup_{k\in\mathbf{Z}_+}\{y\ominus k \mid y\in[0,Rp^j)\}$. It is easy to check that

$$\lim_{j\to-\infty} \mathbf{1}_{S_{R,j}}(x) = 0 \quad \text{for all} \quad x\notin \mathbf{Z}_+.$$

Thus by the dominated convergence theorem from (4.9) we get

$$\lim_{j\to-\infty} ||P_j u|| = 0.$$

In view of (4.8), this implies that $||f||\le\varepsilon$, and thus $\bigcap V_j = \{0\}$.

Proof of Theorem 2. Let a function φ be defined by Walsh-Fourier transform (3.5) and the spaces V_j $(j\in\mathbf{Z}_+)$ are given by (4.7). As before, since $\tilde\varphi(\omega) = m_0(\omega/p)\tilde\varphi(\omega/p)$, we have

$$\varphi(x) = p\sum_{\alpha=0}^{p^n-1} a_\alpha \varphi(px\ominus\alpha),$$

which implies that $V_j\subset V_{j+1}$. On account of Lemma 1, we see that conditions (i) and (iii)–(v) of multiresolution p-analysis are satisfied. By Lemma 2 we have $\bigcap V_j = \{0\}$. Therefore, it remains to confirm that

$$\overline{\bigcup V_j} = L^2(\mathbf{R}_+)$$

or, equvalently,

$$(\bigcup V_j)^\perp = \{0\}. \tag{4.10}$$

Let $f\in(\bigcup V_j)^\perp$. Given an $\varepsilon>0$ we choose $u\in L^1(\mathbf{R}_+)\cap\mathcal{E}(\mathbf{R}_+)$ such that $||f-u||<\varepsilon$. Then for any $j\in\mathbf{Z}_+$ we have

$$||P_jf||^2 = (P_jf,P_jf) = (f,P_jf) = 0$$

and so

$$||P_ju|| = ||P_j(f-u)|| \le ||f-u|| < \varepsilon. \tag{4.11}$$

Choose a positive integer j so large that $\operatorname{supp}\tilde u\subset[0,p^j)$ and $p^{-j}\omega\in[0,p^{-n+1})$ for all $\omega\in\operatorname{supp}\tilde u$. Then we put $g(\omega)=\tilde u(\omega)\tilde\varphi(p^{-j}\omega)$. As the system $\{p^{-j/2}\chi(p^{-j}k,\cdot)\}_{k=0}^\infty$ is an orthonormal bases in $L^2[0,p^j]$, we have

$$\sum_{k\in\mathbf{Z}_+} |c_k(g)|^2 = p^{-j}\int_0^{p^j} |g(\omega)|^2 d\omega, \tag{4.12}$$

where

$$c_k(g) = p^{-j/2} \int_0^{p^j} g(\omega)\overline{\chi(p^{-j}k, \omega)}d\omega.$$

Observing that

$$\int_{\mathbf{R}_+} \varphi(p^j x \ominus k)\overline{\chi(x, \omega)}dx = p^{-j}\tilde{\varphi}(p^{-j}\omega)\overline{\chi(p^{-j}k, \omega)},$$

by the Plancherel relation we get

$$p^{-j/2}(u, \varphi_{j,k}) = p^{-j} \int_0^{p^j} g(\omega)\overline{\chi(p^{-j}k, \omega)}d\omega.$$

Thus, in view of (4.12),

$$||P_j u||^2 = \sum_{k \in \mathbf{Z}_+} |(u, \varphi_{j,k})|^2 = \int_0^{p^j} |\tilde{u}(\omega)\tilde{\varphi}(p^{-j}\omega)|^2 d\omega. \tag{4.13}$$

Since $m_0(\omega) = 1$ on $\Delta_0^{(n)}$ and since $p^{-j}\omega \in [0, p^{-n+1})$ for $\omega \in \operatorname{supp} \tilde{u}$, it follows from (3.5) that $\tilde{\varphi}(p^{-j}\omega) = 1$ for all $\omega \in \operatorname{supp} \tilde{u}$. Furthermore, because $\operatorname{supp} \tilde{u} \subset [0, p^j)$, we obtain from (4.11) and (4.13)

$$\varepsilon > ||P_j u|| = ||\tilde{u}|| = ||u||.$$

Consequently, we conclude

$$||f|| < \varepsilon + ||u|| < 2\varepsilon,$$

which implies (4.10).

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