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Dyadic wavelets and refinable functions on a half-line

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Abstract. For an arbitrary positive integer n refinable functions on the positive half-line $\mathbb{R}_+$ are defined, with masks that are Walsh polynomials of order $2^n - 1$. The Strang-Fix conditions, the partition of unity property, the linear independence, the stability, and the orthonormality of integer translates of a solution of the corresponding refinement equations are studied. Necessary and sufficient conditions ensuring that these solutions generate multiresolution analysis in $L^2(\mathbb{R}_+)$ are deduced. This characterizes all systems of dyadic compactly supported wavelets on $\mathbb{R}_+$ and gives one an algorithm for the construction of such systems. A method for finding estimates for the exponents of regularity of refinable functions is presented, which leads to sharp estimates in the case of small n. In particular, all the dyadic entire compactly supported refinable functions on $\mathbb{R}_+$ are characterized. It is shown that a refinable function is either dyadic entire or has a finite exponent of regularity, which, moreover, has effective upper estimates.

Bibliography: 13 items.

Introduction

Throughout this paper we use the following notation: $\mathbb{R}_+ = [0, +\infty)$ is the positive half-line, $\{w_j\}$ is the Walsh system on $\mathbb{R}_+$, $\oplus$ and $\ominus$ are the dyadic operations in $\mathbb{R}_+$, $\widehat{f}$ is the Walsh-Fourier transform of a function f (see § 1 and also [1], and [2]). As usual, we denote by $\mathbb{N}$ and $\mathbb{Z}_+$ the sets of positive and of non-negative integers, respectively.

Basic facts about orthogonal wavelets and refinable functions on the real line $\mathbb{R}$ can be found in [3]. In this paper, for an arbitrary positive integer n we study solutions φ of the refinement equation

$$\varphi(x) = \sum_{k=0}^{2^n-1} c_k \varphi(2x \ominus k), \qquad x \in \mathbb{R}_+, \tag{0.1}$$

generating multiresolution analyses in $L^2(\mathbb{R}_+)$. The coefficients c_k of equation (0.1) are arbitrary complex numbers. We focus on compactly supported non-trivial solutions $\varphi \in L^2(\mathbb{R}_+)$ of this equation. If such a solution φ exists, then it is unique up to multiplication by a constant and, moreover, $\widehat{\varphi}(0) \neq 0$. In § 2 we show that

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if equation (0.1) possesses a compactly supported solution $\varphi \in L^2(\mathbb{R}_+)$ normalized by the condition $\widehat{\varphi}(0) = 1$, then

$$\sum_{k=0}^{2^n-1} c_k = 2, \quad \operatorname{supp}\varphi \subset [0, 2^{n-1}], \quad \text{and} \quad \widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(2^{-j}\omega), \tag{0.2}$$

where

$$m(\omega) = \frac{1}{2}\sum_{k=0}^{2^n-1} c_k w_k(\omega) \tag{0.3}$$

is a Walsh polynomial called the *mask* of equation (0.1) (or the mask of its solution φ). Furthermore, this solution has the following properties (see § 2 and § 3):

(1) $\widehat{\varphi}(r) = 0$ for all $r \in \mathbb{N}$ (the modified Strang-Fix condition);
(2) $\sum_{k\in\mathbb{Z}_+} \varphi(x \oplus k) = 1$ for almost all $x \in \mathbb{R}_+$ (the partition of unity property);
(3) if the system $\{\varphi(\,\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is orthogonal in $L^2(\mathbb{R}_+)$, then

$$|m(\omega)|^2 + |m(\omega + 1/2)|^2 = 1 \qquad \text{for each } \omega \in [0, 1/2). \tag{0.4}$$

Dyadic intervals of range n are intervals of the following form:

$$I_k^{(n)} = [k2^{-n}, (k+1)2^{-n}), \qquad k \in \mathbb{Z}_+.$$

We recall that for each $0 \leqslant j \leqslant 2^n - 1$ the Walsh function $w_j(x)$ is piecewise constant: on each interval $I_k^{(n)}$ it is either equal to 1 or to -1. Moreover, $w_j(x) = 1$ for $x \in I_0^{(n)}$. We now set

$$b_l = m(\omega) \qquad \text{for} \quad \omega \in I_l^{(n)}, \quad 0 \leqslant l \leqslant 2^n - 1,$$

where m is the mask of refinement equation (0.1). Equalities (0.2) and (0.4) yield

$$b_0 = 1, \qquad |b_l|^2 + |b_{l+2^{n-1}}|^2 = 1, \quad 0 \leqslant l \leqslant 2^{n-1} - 1. \tag{0.5}$$

The coefficients of refinement equation (0.1) are related to the values b_l of mask (0.3) on dyadic intervals by means of the direct and the inverse Walsh transformations:

$$c_k = \frac{1}{2^{n-1}}\sum_{l=0}^{2^n-1} b_l w_l(k2^{-n}), \qquad 0 \leqslant k \leqslant 2^n - 1, \tag{0.6}$$

$$b_l = \frac{1}{2}\sum_{k=0}^{2^n-1} c_k w_k(l2^{-n}), \qquad 0 \leqslant l \leqslant 2^n - 1. \tag{0.7}$$

They can be realized by fast algorithms, which are similar to the classical algorithms of the fast Fourier transformation (see, for instance, [2], Ch. 9). Thus, our choice of the values of a mask (0.3) on the dyadic intervals of range n defines also the coefficients of equation (0.1) for the corresponding function φ.

Definition 1. *Multiresolution analysis* (*MRA*) in $L^2(\mathbb{R}_+)$ is a sequence of closed subspaces $V_j \subset L^2(\mathbb{R}_+)$, $j \in \mathbb{Z}$ such that

(i) $V_j \subset V_{j+1}$ for each $j \in \mathbb{Z}$;

(ii) the union $\bigcup V_j$ is dense in $L^2(\mathbb{R}_+)$, and $\bigcap V_j = \{0\}$;

(iii) $f(\,\cdot\,) \in V_j \Leftrightarrow f(2\,\cdot\,) \in V_{j+1}$ for each $j \in \mathbb{Z}$;

(iv) $f(\,\cdot\,) \in V_0 \Rightarrow f(\,\cdot \oplus k) \in V_0$ for all $k \in \mathbb{Z}_+$;

(v) there is a function $\varphi \in L^2(\mathbb{R}_+)$ such that the system $\{\varphi(\,\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ forms an orthonormal basis in V_0.

The function φ in condition (v) is called a *dyadic scaling function* or a *dyadic refinable function* in $L^2(\mathbb{R}_+)$.

For arbitrary $\varphi \in L^2(\mathbb{R}_+)$ we set

$$\varphi_{jk}(x) = 2^{j/2}\varphi(2^j x \ominus k), \qquad j \in \mathbb{Z}, \quad k \in \mathbb{Z}_+.$$

We say that *a function φ generates MRA in $L^2(\mathbb{R}_+)$* if the system $\{\varphi(\,\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$ and, in addition, the family of subspaces

$$V_j = \operatorname{clos}_{L^2(\mathbb{R}_+)} \operatorname{span}\{\varphi_{jk} \mid k \in \mathbb{Z}_+\}, \qquad j \in \mathbb{Z}, \tag{0.8}$$

is MRA in $L^2(\mathbb{R}_+)$. If a function φ generates MRA in $L^2(\mathbb{R}_+)$, then it is a dyadic refinable function in $L^2(\mathbb{R}_+)$. Each dyadic refinable function φ defines by means of a routine procedure (see [4]–[6]) a *dyadic wavelet* ψ on $\mathbb{R}_+$ such that the functions

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x \ominus k), \qquad j \in \mathbb{Z}, \quad k \in \mathbb{Z}_+,$$

form an orthonormal basis in $L^2(\mathbb{R}_+)$.

Example 1. In the case of $n = 1$ and $c_0 = c_1 = 1$ the solution of (0.1) is the Haar function $\varphi = \chi_{[0,1)}$ (throughout this paper we denote by χ_E the indicator function of the set E). In this case

$$m(\omega) = \begin{cases} 1 & \text{for } \omega \in [0, 1/2), \\ 0 & \text{for } \omega \in [1/2, 1), \end{cases}$$

$$\psi(x) = \begin{cases} 1 & \text{for } x \in [0, 1/2), \\ -1 & \text{for } x \in [1/2, 1), \\ 0 & \text{for } x \in \mathbb{R}_+ \setminus [0, 1). \end{cases}$$

The corresponding wavelets system $\{\psi_{jk}\}$ is the classical Haar system.

Example 2. The Lang refinable function [4] occurs in the case $n = 2$ for the mask

$$m(\omega) = \begin{cases} 1 & \text{for } \omega \in [0, 1/4), \\ a & \text{for } \omega \in [1/4, 1/2), \\ 0 & \text{for } \omega \in [1/2, 3/4), \\ b & \text{for } \omega \in [3/4, 1), \end{cases}$$

where $0 < |a| < 1$, $|b| = \sqrt{1 - |a|^2}$. The function φ satisfies the equation

$$\varphi(x) = \sum_{k=0}^{3} c_k \varphi(2x \ominus k)$$

with coefficients

$$c_0 = \frac{1+a+b}{2}, \qquad c_1 = \frac{1+a-b}{2}, \qquad c_2 = \frac{1-a-b}{2}, \qquad c_3 = \frac{1-a+b}{2}.$$

This function generates MRA in $L^2(\mathbb{R}_+)$, possesses the following self-similarity property:

$$\varphi(x) = \begin{cases} \dfrac{1+a-b}{2} + b\varphi(2x) & \text{for } 0 \leqslant x < 1, \\ \dfrac{1-a+b}{2} - b\varphi(2x-2) & \text{for } 1 \leqslant x \leqslant 2, \end{cases}$$

and is represented by a lacunary Walsh series:

$$\varphi(x) = \frac{1}{2}\chi_{[0,1)}\left(\frac{x}{2}\right)\left(1 + a\sum_{j=0}^{\infty} b^j w_{2^{j+1}-1}\left(\frac{x}{2}\right)\right), \qquad x \in \mathbb{R}_+. \tag{0.9}$$

Furthermore, for $|b| < 1/2$ the corresponding wavelets system $\{\psi_{jk}\}$ is an unconditional basis in all the spaces $L^q(\mathbb{R}_+)$, $1 < q < \infty$ (see [5]).

We now denote by $\ominus_p$ the operation of subtraction modulo p in $\mathbb{R}_+$. In [6] for all $p, n \geqslant 2$ the author finds coefficients c_k, $0 \leqslant k \leqslant p^n - 1$ such that the solution φ of the equation

$$\varphi(x) = \sum_{k=0}^{p^n-1} c_k \varphi(px \ominus_p k), \qquad x \in \mathbb{R}_+, \tag{0.10}$$

possesses the following properties:
(1) φ is the sum of a lacunary Walsh series;
(2) the system $\{\varphi(\,\cdot \ominus_p k) \mid k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$;
(3) $\operatorname{supp}\varphi \subset [0, p^{n-1}]$;
(4) φ generates p-multiresolution analysis in $L^2(\mathbb{R}_+)$.

For the calculation of the coefficients of equation (0.10) one chooses $p^n - p$ complex parameters satisfying a certain 'orthogonality condition', complements them by $p-1$ zeros, and applies the fast Vilenkin–Chrestenson transform. In [7] the author constructs similar refinable functions and the corresponding wavelets on the locally compact Abelian group G_p of sequences $x = (x_j) = (\dots, 0, 0, x_k, x_{k+1}, x_{k+2}, \dots)$, where $x_j \in \{0, 1, \dots, p-1\}$ for $j \in \mathbb{Z}$ and $x_j = 0$ for $j < k = k(x)$. The group operation in G_p is coordinatewise addition modulo p, and the topology corresponds to the complete system of neighbourhoods of zero

$$U_l = \{(x_j) \in G_p : x_j = 0,\ j \leqslant l\}, \qquad l \in \mathbb{Z}.$$

In the case $p = 2$ the subgroup U_0 is isomorphic to the *Cantor dyadic group*, that is, the topological direct product of countably many cyclic groups of the second order equipped with the discrete topology. Basic facts and methods of the theory of harmonic analysis on G_2 (the additive group of the dyadic field $\mathbb{F}$) can be found in the monograph [2]. Dyadic wavelets on this group were studied in [4], [5].

The results of this paper concern mainly the following four problems:

1. Find necessary and sufficient conditions in order that solutions of functional equation (0.1) generate multiresolution analysis in $L^2(\mathbb{R}_+)$.

2. Derive conditions in order that the system $\{\varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$, where φ is a solution of (0.1), be linearly independent, stable, or orthonormal in $L^2(\mathbb{R}_+)$.

3. Estimate the regularity of solutions of equation (0.1).

4. Derive necessary and sufficient conditions in order that solutions of (0.1) be infinitely smooth or dyadic entire.

We shall indicate some crucial differences between our results in this paper and the corresponding results of the classical wavelet analysis on $\mathbb{R}$ (see Remark 2). Some of our results extend in a natural fashion to wavelets and refinable functions on the group G_p, as also to solutions of equation (0.10) on the half-line $\mathbb{R}_+$.

We start with some definitions. The family $\{[0, 2^{-j}) \mid j \in \mathbb{Z}\}$ forms a fundamental system of the dyadic topology in $\mathbb{R}_+$ (see, for example, [2], § 1.3). A subset E of $\mathbb{R}_+$ that is compact in the dyadic topology is said to be *W-compact*. It is easy to see that the union of a finite family of dyadic intervals is W-compact. A W-compact set E is said to be *congruent to* $[0,1)$ *modulo* $\mathbb{Z}_+$ if its Lebesgue measure is 1 and for each $x \in [0,1)$ there exists $k \in \mathbb{Z}_+$ such that $x \oplus k \in E$. We say that a Walsh polynomial m satisfies the *modified Cohen criterion* if there exists a W-compact subset E of $\mathbb{R}_+$ congruent to $[0,1)$ modulo $\mathbb{Z}_+$ and containing a neighbourhood of zero such that

$$\inf_{j\in\mathbb{N}} \inf_{\omega\in E} |m(2^{-j}\omega)| > 0. \tag{0.11}$$

For an arbitrary set $M \subset [0,1)$ we set

$$TM := \frac{1}{2}M \cup \left(\frac{1}{2} + \frac{1}{2}M\right), \tag{0.12}$$

where $\alpha + \beta M := \{\alpha + \beta x \mid x \in M\}$.

Definition 2. Let m be the mask of refinement equation (0.1). A set $M \subset [0,1)$ is said to be *blocked* (for the mask m) if it is a union of dyadic intervals of range $n-1$, does not contain the interval $[0, 2^{-n+1})$, and possesses the property

$$TM \subset M \cup \operatorname{Null} m,$$

where $\operatorname{Null} m$ is the zero set of the mask m on $[0,1)$.

For a compactly supported L^2-solution φ of equation (0.1) such that $\widehat{\varphi}(0) = 1$ it follows by the orthonormality of the system $\{\varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ in $L^2(\mathbb{R}_+)$ that

$$m(0) = 1, \qquad |m(\omega)|^2 + |m(\omega + 1/2)|^2 = 1 \quad \text{for each } \omega \in [0, 1/2). \tag{0.13}$$

In § 4 we establish the converse result: if a mask m of a compactly supported L^2-solution φ of equation (0.1) satisfies (0.13) and one of the following equivalent conditions:

(1) m has no blocked sets;

(2) m satisfies the modified Cohen criterion,

then φ generates MRA in $L^2(\mathbb{R}_+)$ (and, therefore, the system $\{\varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$).

Computing the quantities b_l, $0 \leqslant l \leqslant 2^n - 1$ for a fixed mask m by formula (0.7) one can write equalities (0.13) in the form (0.5). We point out that since $m(\omega) \equiv 1$

on $I_0^{(n)}$, it is sufficient for the verification of condition (0.11) to find an integer j_0 such that $E/2^{j_0} \subset I_0^{(n)}$ and to verify that m does not vanish on the sets $E/2, \dots, E/2^{j_0-1}$. If $m(\omega) \neq 0$ on $[0,1/2)$, then (0.11) holds for $E=[0,1)$ (see Examples 1, 2, 5). In the next example we define a one-parameter family of dyadic entire refinable functions.

Example 3. Let $n=3$ and let

$$m(\omega)=\begin{cases} 1 & \text{if } \omega\in[0,1/4)\cup[3/8,1/2),\\ b & \text{if } \omega\in[1/4,3/8),\\ 0 & \text{if } \omega\in[1/2,3/4)\cup[7/8,1),\\ \beta, & \text{if } \omega\in[3/4,7/8), \end{cases} \tag{0.14}$$

where $0\leqslant |b|<1$, $|\beta|=\sqrt{1-|b|^2}$. Then (0.2) yields

$$\widehat{\varphi}(\omega)=\chi_{[0,1/2)}(\omega)+b\chi_{[1/2,3/4)}(\omega)+\chi_{[3/4,1)}(\omega)+\beta\chi_{[3/2,7/4)}(\omega).$$

Computing the inverse Walsh-Fourier transform we obtain the refinable function

$$\varphi(x)=\frac{1}{4}\chi_{[0,4)}(x)\biggl[1+w_1\biggl(\frac{x}{4}\biggr)+bw_2\biggl(\frac{x}{4}\biggr)+w_3\biggl(\frac{x}{4}\biggr)+\beta w_6\biggl(\frac{x}{4}\biggr)\biggr], \tag{0.15}$$

the mask of which satisfies (0.11) for $E=[0,1/2)\cup[3/4,1)\cup[3/2,7/4)$. Function (0.15) generates MRA in $L^2(\mathbb{R}_+)$ and satisfies the equation

$$\varphi(x)=\sum_{k=0}^{7}c_k\varphi(2x\ominus k)$$

with coefficients

$$c_0=\frac{3+b+\beta}{4},\qquad c_1=\frac{3+b-\beta}{4},\qquad c_2=c_6=\frac{1-b-\beta}{4},$$
$$c_3=c_7=\frac{1-b+\beta}{4},\qquad c_4=\frac{-1+b+\beta}{4},\qquad c_5=\frac{-1+b-\beta}{4}.$$

These representations of the coefficients follow from (0.6) and (0.14). For $b=0$ function (0.15) was considered in [6], Example 4.

In § 5 we obtain expansions of refinable functions in lacunary Walsh series and elaborate a method for the computation of the regularity of these functions (which produces precise values in the case of small n). Similar results for the locally compact Abelian group G_p can be found in the recent paper [7]. In § 6 we derive a criterion for refinable functions to be dyadic entire, and in § 7 we prove that a refinable function is either dyadic entire or has a finite exponent of regularity with an effective upper estimate.

§ 1. Preliminary facts and results

The Walsh system $\{w_n \mid n\in\mathbb{Z}_+\}$ on $\mathbb{R}_+$ is defined as follows:

$$w_0(x)\equiv 1,\qquad w_n(x)=\prod_{j=0}^{k}(w_1(2^jx))^{\nu_j},\quad n\in\mathbb{N},\qquad x\in\mathbb{R}_+,$$

where the ν_j are the coefficients of the decomposition

$$n = \sum_{j=0}^{k} \nu_j 2^j, \qquad \nu_j \in \{0,1\}, \quad \nu_k = 1, \quad k = k(n),$$

and the function $w_1(x)$ is defined on $[0,1)$ by the formula

$$w_1(x) = \begin{cases} 1, & \text{if } x \in [0,1/2), \\ -1, & \text{if } x \in [1/2,1), \end{cases}$$

and is extended to $\mathbb{R}_+$ by periodicity: $w_1(x+1) = w_1(x)$ for all $x \in \mathbb{R}_+$.

Walsh polynomials are finite linear combinations of the Walsh functions. An arbitrary Walsh polynomial of order n can be written in the following form:

$$w(x) = \sum_{j=0}^{n} c_j w_j(x),$$

where the c_j are complex coefficients. For more information about the properties of Walsh polynomials and their role in the dyadic harmonic analysis see, for example, [1], [2].

We shall denote the integer and the fractional parts of a number $x \in \mathbb{R}_+$ by $[x]$ and $\{x\}$ respectively.

For $x \in \mathbb{R}_+$ and $j \in \mathbb{N}$ we define the numbers $x_j, x_{-j} \in \{0,1\}$ as follows:

$$x_j = [2^j x] \pmod 2, \qquad x_{-j} = [2^{1-j} x] \pmod 2. \tag{1.1}$$

They are the digits of the binary expansion

$$x = \sum_{j<0} x_j 2^{-j-1} + \sum_{j>0} x_j 2^{-j}$$

(for dyadic x we obtain an expansion with finitely many non-zero terms).

For fixed $x, y \in \mathbb{R}_+$ we set

$$x \oplus y = \sum_{j<0} |x_j - y_j| 2^{-j-1} + \sum_{j>0} |x_j - y_j| 2^{-j},$$

where x_j, y_j are defined in (1.1). By definition $x \ominus y = x \oplus y$ (because $x \oplus x = 0$). The binary operation $\oplus$ identifies $\mathbb{R}_+$ with the group G_2 and is useful in the study of dyadic Hardy classes and for the construction of algorithms in signal processing (see [1], [2]). We point out that this identification associates Haar measure in G_2 with Lebesgue measure on $\mathbb{R}_+$, and the characters of the group G_2 with generalized Walsh functions.

A function $f\colon \mathbb{R}_+ \to \mathbb{C}$ is said to be *W-continuous at a point* $x \in \mathbb{R}_+$ if

$$\sup_{0 \leqslant h < 1/2^n} |f(x \oplus h) - f(x)| \to 0 \qquad \text{as } n \to \infty.$$

A function f is *W-continuous* if it is W-continuous at each point of $\mathbb{R}_+$. A continuous (in the usual sense) function on $\mathbb{R}_+$ is also W-continuous. Walsh polynomials are W-continuous (see, for example, [2], § 1.3).

For $x, \omega \in \mathbb{R}_+$ we set

$$\chi(x,\omega) = (-1)^{\sigma(x,\omega)}, \qquad \text{where} \quad \sigma(x,\omega) = \sum_{j=1}^{\infty} x_j\omega_{-j} + x_{-j}\omega_j$$

and x_j, ω_j are defined by (1.1). Observe that for each positive integer n we have

$$\chi(x, 2^{-n}l) = \chi(2^{-n}x, l) = w_l(2^{-n}x) \qquad \text{for all} \quad x \in [0, 2^n), \quad l \in \mathbb{Z}_+.$$

The Walsh-Fourier transform of the function $f \in L^1(\mathbb{R}_+)$ is defined as follows:

$$\widehat{f}(\omega) = \int_{\mathbb{R}_+} f(x)\chi(x,\omega)\,dx.$$

If $f \in L^2(\mathbb{R}_+)$ and

$$J_a f(\omega) = \int_0^a f(x)\chi(x,\omega)\,dx, \qquad a > 0,$$

then $\widehat{f}$ is the limit of $J_a f$ in $L^2(\mathbb{R}_+)$ as $a \to \infty$.

The properties of the Walsh-Fourier transformation are quite similar to the properties of the classical Fourier transformation (see, for example, [1], Ch. 6 or [2], Ch. 9). We list some of them below.

Proposition 1. *The following properties hold*:

(a) *if $f \in L^1(\mathbb{R}_+)$, then $\widehat{f}$ is a W-continuous function and $\widehat{f}(\omega) \to 0$ as $\omega \to \infty$;*

(b) *the inversion formula*

$$f(x) = \int_{\mathbb{R}_+} \widehat{f}(\omega)\chi(x,\omega)\,d\omega;$$

holds for each $x \in \mathbb{R}_+$, provided that both f and $\widehat{f}$ belong to $L^1(\mathbb{R}_+)$ and f is W-continuous.

(c) *if $f \in L^2(\mathbb{R}_+)$, then $\widehat{f} \in L^2(\mathbb{R}_+)$ and*

$$\|\widehat{f}\|_{L^2(\mathbb{R}_+)} = \|f\|_{L^2(\mathbb{R}_+)}.$$

In what follows $\mathscr{E}_n$ is the space of *dyadic entire functions of order n*, that is, of functions defined in $\mathbb{R}_+$ and constant on all intervals of range n. For $f \in \mathscr{E}_n$ one has

$$f(x) = \sum_{k=0}^{\infty} f(2^{-n}k)\chi_{I_k^{(n)}}(x), \qquad x \in \mathbb{R}_+.$$

Clearly, each Walsh polynomial of order $2^n - 1$ belongs to $\mathscr{E}_n$. The set $\mathscr{E}$ of dyadic entire functions on $\mathbb{R}_+$ is the union of all the spaces $\mathscr{E}_n$. The refinable functions in Examples 1 and 3 belong to $\mathscr{E}$.

Proposition 2. *The following properties hold*:

(a) *if* $\operatorname{supp}\widehat{f}\subset[0,2^n]$, *then* $f\in\mathscr{E}_n$, *provided that both* f *and* $\widehat{f}$ *belong to* $L^1(\mathbb{R}_+)$ *and* f *is* W*-continuous*;

(b) *if* $f\in L^1(\mathbb{R}_+)\cap\mathscr{E}_n$, *then* $\operatorname{supp}\widehat{f}\subset[0,2^n]$;

(c) *if* $f\in L^1(\mathbb{R}_+)$ *and* $\operatorname{supp}f\subset[0,2^n]$, *then* $\widehat{f}\in\mathscr{E}_n$; *in a similar way, if* $g\in L^1(\mathbb{R}_+)$ *and* $\operatorname{supp}g\subset[0,2^n]$, *then the inverse Walsh-Fourier transform of the function* g *belongs to* $\mathscr{E}_n$;

(d) *if* $\widehat{f}\in\mathscr{E}_n$, *then* $\operatorname{supp}f\subset[0,2^n]$, *provided that* $f\in L^1(\mathbb{R}_+)$ *and* f *is* W*-continuous.*

Proofs of all these properties can be found in [1], § 6.2. We also require the following two propositions from [6].

Proposition 3. *Let* $\varphi\in L^2(\mathbb{R}_+)$. *Then the system* $\{\varphi(\,\cdot\ominus k)\mid k\in\mathbb{Z}_+\}$ *is orthonormal in* $L^2(\mathbb{R}_+)$ *if and only if*

$$\sum_{l\in\mathbb{Z}_+}|\widehat{\varphi}(\omega\oplus l)|^2=1 \qquad \text{for almost all} \quad \omega\in\mathbb{R}_+. \tag{1.2}$$

Proposition 4. *Let* $\{V_j\}$ *be a family of subspaces defined by formula* (0.8) *with a fixed function* $\varphi\in L^2(\mathbb{R}_+)$. *If the system*

$$\{\varphi(\,\cdot\ominus k)\mid k\in\mathbb{Z}_+\}$$

forms an orthonormal basis in V_0, *then* $\bigcap V_j=\{0\}$.

Results similar to Propositions 3, 4 and useful in the construction of wavelets in $L^2(\mathbb{R})$ are well known (see [3], §§ 5.1, 5.3).

§ 2. The uniqueness of the solution, the Strang-Fix condition, and the partition of unity property

Let $\varphi\in L^2(\mathbb{R}_+)$ be a solution of equation (0.1). Using the Walsh-Fourier transform we obtain

$$\widehat{\varphi}(\omega)=m\biggl(\frac{\omega}{2}\biggr)\widehat{\varphi}\biggl(\frac{\omega}{2}\biggr), \tag{2.1}$$

where

$$m(\omega)=\frac{1}{2}\sum_{k=0}^{2^n-1}c_k w_k(\omega) \tag{2.2}$$

is the mask of refinement equation (0.1).

Theorem 1. *If equation* (0.1) *has a compactly supported solution* $\varphi\in L^2(\mathbb{R}_+)$ *such that* $\widehat{\varphi}(0)=1$, *then*

$$\sum_{k=0}^{2^n-1}c_k=2 \quad \text{and} \quad \operatorname{supp}\varphi\subset[0,2^{n-1}]. \tag{2.3}$$

This solution is unique, is given by the formula

$$\widehat{\varphi}(\omega)=\prod_{j=1}^{\infty},m(2^{-j}\omega) \tag{2.4}$$

and possesses the following properties:

(1) $\widehat{\varphi}(r) = 0$ *for all* $r \in \mathbb{N}$ (*the modified Strang-Fix condition*);

(2) $\sum_{k\in\mathbb{Z}_+} \varphi(x \oplus k) = 1$ *for almost all* $x \in \mathbb{R}_+$ (*the partition of unity property*).

Proof. Assume that equation (0.1) possesses a solution $\varphi \in L^2(\mathbb{R}_+)$ with compact support such that $\widehat{\varphi}(0) = 1$. Substituting $\omega = 0$ in (2.1) we obtain $m(0) = 1$, therefore $\sum_{k=0}^{2^n-1} c_k = 2$. Further, let j be the greatest integer such that φ does not vanish on a positive-measure subset of the interval $[j-1, j]$. Assume that $j \geqslant 2^{n-1}+1$ and consider an arbitrary dyadic $x \in [j-1, j]$. If

$$[2x] = l \pmod{2^n},$$

where $l \in \{1, 2, \dots, 2^n-1\}$, then $2x \geqslant 2^n + l$ and for each $k \in \{0, 1, \dots, 2^n-1\}$ we have

$$2x \ominus k \geqslant 2x - l \geqslant 2^n.$$

Moreover, since $\{2x\} > 0$, it follows that $2x \ominus k > 2^n$. Combining this with (0.1) we conclude that $j \geqslant 2^n+1$; otherwise $\varphi(2x \ominus k) = 0$ for almost all $x \in [j-1, j]$ and for each k. Applying now (0.1) we obtain $\varphi(x) = 0$ a.e. on $[j-1, j]$, which contradicts our choice of j. Furthermore, since $2x$ is not an integer and $x \geqslant j-1$, it follows that for each $k \in \{0, 1, \dots, 2^n-1\}$

$$2x \ominus k \geqslant 2x - (2^n-1) > (2j-2) - (2^n-1) \geqslant j$$

(we use here the inequality $j \geqslant 2^n+1$). Arguing as above we obtain $\varphi(x) = 0$ a.e. on $[j-1, j]$. Thus, $j \leqslant 2^{n-1}$, therefore $\operatorname{supp}\varphi \subset [0, 2^{n-1}]$, and we arrive at (2.3).

We now claim that the Walsh-Fourier transform $\widehat{\varphi}$ satisfies (2.4). First, φ belongs to $L^1(\mathbb{R}_+)$ because it belongs to $L^2(\mathbb{R}_+)$ and has a compact support. Applying now (2.3) and Proposition 2(c) we obtain $\widehat{\varphi} \in \mathscr{E}_{n-1}$. Using the condition $\widehat{\varphi}(0) = 1$ we see that $\widehat{\varphi}(\omega) = 1$ for all $\omega \in [0, 2^{1-n})$. On the other hand, $m(\omega) = 1$ for $\omega \in [0, 2^{1-n})$. Hence, for each $\omega \in [0, 2^r)$ (r is a positive integer) one has

$$\widehat{\varphi}(\omega) = \widehat{\varphi}(2^{-r-n}\omega)\prod_{k=1}^{r+n} m(2^{-k}\omega) = \prod_{k=1}^{\infty} m(2^{-k}\omega),$$

which completes the proof of (2.4) and of the uniqueness of φ.

We observe that for each $r \in \mathbb{N}$ we have

$$\widehat{\varphi}(r) = \widehat{\varphi}(r)\prod_{s=0}^{j-1} m(2^s r) = \widehat{\varphi}(2^j r) \to 0$$

as $j \to \infty$ (since $\widehat{\varphi} \in L^1(\mathbb{R}_+)$ and $m(2^s r) = 1$ because $m(0) = 1$ and m is periodic). This means that $\widehat{\varphi}(r) = 0$. By Poisson's summation formula we obtain

$$\sum_{k=0}^{\infty} \varphi(x \oplus k) = \sum_{r=0}^{\infty} \widehat{\varphi}(r) w_r(x)$$

(the equality holds almost everywhere in Lebesgue measure). Since $\widehat{\varphi}(r) = \delta_{0r}$, it follows that

$$\sum_{k=0}^{\infty} \varphi(x \oplus k) = \widehat{\varphi}(0) w_0(x) = 1.$$

The proof of the theorem is now complete.

§ 3. Linear independence and stability in $L^q(\mathbb{R}_+)$

A function $f \in L^q(\mathbb{R}_+)$, $1 \leqslant q \leqslant \infty$ is *q-stable* if there exist positive constants A_q and B_q such that

$$A_q \|a\|_{\ell^q} \leqslant \biggl\| \sum_{k \in \mathbb{Z}_+} a_k f(\,\cdot \ominus k) \biggr\|_{L^q(\mathbb{R}_+)} \leqslant B_q \|a\|_{\ell^q} \tag{3.1}$$

for each sequence $a = \{a_k\} \in \ell^q$. In particular, a function $f \in L^2(\mathbb{R}_+)$ is 2-stable if and only if $\{f(\,\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is a Riesz system in $L^2(\mathbb{R}_+)$ (for more information about Riesz bases and systems see, for example, [8]). We say that a function $f \colon \mathbb{R}_+ \to \mathbb{C}$ has a *periodic zero* at a point $x \in \mathbb{R}_+$ if $f(x \oplus k) = 0$ for all $k \in \mathbb{Z}_+$.

Theorem 2. *For a compactly supported function $f \in L^q(\mathbb{R}_+)$, $1 \leqslant q \leqslant \infty$ the following properties are equivalent*:

(1) *f is q-stable*;
(2) *the system $\{f(\,\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is linearly independent*;
(3) *the Walsh-Fourier transform of f does not have periodic zeros.*

Proof. Since f has compact support and belongs to $L^q(\mathbb{R}_+)$ with $q \geqslant 1$, it follows that f also belongs to $L^1(\mathbb{R}_+)$. Let $\operatorname{supp} f \subset [0, 2^{n-1}]$ for some positive integer n; then $\widehat{f} \in \mathscr{E}_{n-1}$ (Proposition 2(c)). We observe that the linear independence of the system $\{f(\,\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is equivalent to that of the finite system $f(\,\cdot \ominus k)$, $k \in \{0, 1, \dots, 2^{n-1} - 1\}$, because the other functions have supports disjoint from the interval $[0, 2^{n-1}]$. If there exist $a_0, \dots, a_{2^{n-1}-1}$ such that

$$\sum_{k=0}^{2^{n-1}-1} a_k f(\,\cdot \ominus k) = 0 \quad \text{and} \quad |a_0| + \dots + |a_{2^{n-1}-1}| > 0, \tag{3.2}$$

then the sequence $(a_0, \dots, a_{2^{n-1}-1}, 0, 0, \dots)$ fails the lower bound in (3.1). Conversely, if f is not q-stable, then the function

$$F(a) = \biggl\| \sum_{k=0}^{2^{n-1}-1} a_k f(\,\cdot \ominus k) \biggr\|_{L^q(\mathbb{R}_+)}$$

takes arbitrarily small values on the sphere

$$S = \biggl\{ a = (a_0, \dots, a_{2^{n-1}-1}) \biggm| \sum_{k=0}^{2^{n-1}-1} |a_k| = 1 \biggr\}.$$

Indeed, the right hand side of (3.1) always holds for compactly supported functions: if $\operatorname{supp} f \subset [0, 2^{n-1})$, then one can set $B_q = 2^{(n-1)(1-1/q)} \|f\|_{\ell_q}$. This follows immediately from Hölder's inequality in the ℓ_q-metric. Combining the linearity of F and the compactness of a unit ball in a finite-dimensional space we conclude that there exists a point $a \in S$ such that $F(a) = 0$, which means the linear dependence of the integer translates. Thus, (1) $\Leftrightarrow$ (2). Further, if some set $a = (a_0, \dots, a_{2^{n-1}-1})$ satisfies conditions (3.2), then using the Walsh-Fourier transformation we obtain

$$\widehat{f}(\omega) \sum_{k=0}^{2^{n-1}-1} a_k w_k(\omega) = 0 \qquad \text{for almost all} \quad \omega \in \mathbb{R}_+.$$

The Walsh polynomial $w(\omega) = \sum_{k=0}^{2^{n-1}-1} a_k w_k(\omega)$ is not identically equal to zero; hence there exists a dyadic interval I of range $n-1$ such that $w(I+r) \neq 0$, $r \in \mathbb{Z}_+$. By the periodicity of w one can assume that I lies in the interval $[0,1)$. Since $\widehat{f} \in \mathscr{E}_{n-1}$, it follows that (3.2) holds if and only if there exists a dyadic interval $I \subset [0,1)$ of range $n-1$ such that $\widehat{f}(I+r) = 0$ for all $r \in \mathbb{Z}_+$. Thus, (2) $\Leftrightarrow$ (3), which completes the proof of Theorem 2.

Corollary 1. *If a function f is compactly supported and q-stable, then it is p-stable for all $p \in [1,q]$.*

Indeed, a q-stable compactly supported function f belongs to all the spaces $L^p(\mathbb{R}_+)$, $1 \leqslant p \leqslant q$.

Remark 1. In the proof of Theorem 2 we have actually established the following result: if the integer translates of a compactly supported function $f \in L^q(\mathbb{R}_+)$, $1 \leqslant q \leqslant \infty$, are linearly dependent, then there exists a dyadic interval $I \subset [0,1)$ consisting entirely of periodic zeros of the Walsh-Fourier transform $\widehat{f}$. Furthermore, if $\operatorname{supp} f \subset [0, 2^{n-1}]$, then I has range $n-1$. Each periodic zero $\omega_0 \in [0,1)$ of $\widehat{f}$ lies in such an interval I.

Thus, the q-stability of a function f, $1 \leqslant q \leqslant \infty$, is equivalent to the linear independence of its integer translates, and to the absence of periodic zeros of its Walsh-Fourier transform $\widehat{f}$. For this reason we shall say in what follows that a function is stable without specifying the value of the parameter q.

We shall now deduce conditions for refinement equation (0.1) to have a stable solution. To this end we require the concept of blocked set (Definition 2) and the operator T defined in (0.12). We shall occasionally denote dyadic subintervals $I_k^{(j)} = [2^{-j}k, 2^{-j}(k+1))$ of $[0,1)$ by $I_k^{(j)} = I_{d_1 \dots d_j}$, where $0.d_1 \dots d_j = 2^{-j}k$. If a set M is blocked for a mask m, then for each dyadic interval $I_{d_1 \dots d_{n-1}} \subset M$ each of the intervals $I_{0d_1 \dots d_j}$ and $I_{1d_1 \dots d_j}$ lies in M or in $\operatorname{Null} m$.

Proposition 5. *Let φ be a compactly supported solution of refinement equation (0.1), $\varphi \in L^q(\mathbb{R}_+)$ for some $1 \leqslant q \leqslant \infty$, and $\widehat{\varphi}(0) = 1$. Then the function φ is not stable if and only if the corresponding mask m possesses a blocked set.*

Proof. By Theorem 1, $\operatorname{supp} \varphi \subset [0, 2^{n-1}]$, therefore $\widehat{\varphi} \in \mathscr{E}_{n-1}$. If φ is not stable, then the set of all periodic zeros of the function $\widehat{\varphi}$ on $[0,1)$ is a blocked set for m. Indeed, the set

$$M_0 = \bigl\{\omega \in [0,1) \mid \widehat{\varphi}(\omega + k) = 0 \text{ for all } k \in \mathbb{Z}_+\bigr\}.$$

is a union of several dyadic intervals of range $n-1$ (Remark 1). Since $\widehat{\varphi}(0)=1$, it follows that M_0 does not contain the interval $[0, 2^{-n+1})$. Furthermore, if $\omega \in M_0$, then by formula (2.1)

$$m\left(\frac{\omega}{2}+\frac{k}{2}\right)\widehat{\varphi}\left(\frac{\omega}{2}+\frac{k}{2}\right)=0 \qquad \text{for all } k \in \mathbb{Z}_+$$

and therefore the numbers $\omega/2$, $\omega/2+1/2$ belong to either M_0 or $\operatorname{Null} m$.

Conversely, if the mask m has a blocked set M, then each $\omega \in M$ possesses the property $\widehat{\varphi}(\omega+k)=0$, $k \in \mathbb{Z}_+$. Hence $\widehat{\varphi}$ has a periodic zero and by Theorem 2 the function φ is not stable. Indeed, assume that there exists $\omega \in M$ such that $\widehat{\varphi}(\omega+k) \neq 0$ for some k. Consider sufficiently large j such that $2^{-j}(\omega+k) < 2^{1-n}$. For each $r \in \{0,1,\dots,j\}$ we denote by x_r the fractional part of the number $2^{-r}(\omega+k)$. Obviously, $x_0=\omega$ and $x_j = 2^{-j}(\omega+k)$. Thus, we have

$$\widehat{\varphi}(\omega+k)=\widehat{\varphi}(2^{-j}(\omega+k))\prod_{r=1}^{j} m(2^{-r}(\omega+k))=\widehat{\varphi}(x_j)\prod_{r=1}^{j} m(x_r). \tag{3.3}$$

It is easy to show that if $x_r \in M$, then $x_{r+1} \in TM$ and therefore x_{r+1} belongs to either $\operatorname{Null} m$ or M. We also point out that $x_r \notin \operatorname{Null} m$, for otherwise it follows by (3.3) that $\widehat{\varphi}(\omega+k)=0$. Thus, if $x_r \in M$, then $x_{r+1} \in M$. On the other hand, since $x_0=\omega \in M$, it follows that $x_r \in M$ for all $1 \leqslant r \leqslant j$. This is impossible because $x_j \notin M$. In fact, $x_j = 2^{-j}(\omega+k) < 2^{1-n}$; however, M does not contain points of the interval $[0, 2^{1-n})$. This contradiction completes the proof of Proposition 5.

Proposition 5 reduces the stability problem for a compactly supported refinable function to the verification of some combinatorial fact, which can be verified, at least theoretically, in finite time by mere brute force. In practice, however, this procedure can take too long for large n because it requires about $2^{2^{n-1}}$ operations. It could be more convenient to use necessary or sufficient conditions for stability. We now formulate several such conditions. We call a point ω a *symmetric zero* of a mask m if $m(\omega)=m(\omega+1/2)=0$.

Corollary 2. *If a mask m possesses a symmetric zero, then the solution φ of refinement equation (0.1) is not stable.*

Indeed, if $\omega = 0.d_1 d_2 \dots d_n \dots$ is a symmetric zero, then the dyadic interval $I_{d_2 \dots d_n}$ is a blocked set.

Corollary 3. *If $m(1/2-1/2^n)=0$, then the solution φ of refinement equation (0.1) is not stable.*

Indeed, if $m(1/2-1/2^n)=0$, then the interval $[1-2^{1-n}, 1)$ is a blocked set.

Corollary 4. *If a mask m has no symmetric zeros and does not vanish at the point $\omega=1/2-1/2^n$ or on the interval $[0,1/4)$, then the solution φ of refinement equation (0.1) is stable.*

The proof is left to the reader.

Another useful consequence of Theorems 1 and 2 provides necessary conditions for the existence of stable L^2-solutions.

Corollary 5. *If an L^2-solution of refinement equation (0.1) is stable, then*

$$m\left(\frac{1}{2}\right) = 0 \quad \textit{and} \quad \sum_{k=0}^{2^{n-1}-1} c_{2k} = \sum_{k=0}^{2^{n-1}-1} c_{2k+1} = 1. \tag{3.4}$$

Proof. Assume that $m(1/2) \neq 0$; then by the Strang-Fix condition (Theorem 1), for each $k \in \mathbb{Z}_+$ we obtain

$$0 = \widehat{\varphi}(2k+1) = m\left(k + \frac{1}{2}\right)\widehat{\varphi}\left(k + \frac{1}{2}\right) = m\left(\frac{1}{2}\right)\widehat{\varphi}\left(k + \frac{1}{2}\right)$$

and therefore $\widehat{\varphi}(k + 1/2) = 0$. This means that $1/2$ is a periodic zero of $\widehat{\varphi}$, which contradicts the stability (Theorem 2). Hence $m(1/2) = 0$ and (3.4) holds. This proves Corollary 5.

How can one determine from the coefficients of the equation (or by its mask) whether the solution belongs to L^q? It is well known that in the classical situation (for refinable functions on the real line $\mathbb{R}$) and in our case (on the half-line $\mathbb{R}_+$) alike this problem is not necessarily effectively soluble. For example, it is still unknown if there exists a practically applicable criterion or an algorithm deciding whether the solution of a refinement equation is continuous. The same situation with the space $L^1(\mathbb{R})$ is similar (see, for example, [9], [10]). There exists a criterion for $L^2(\mathbb{R})$ in terms of the spectral radius of a certain matrix constructed from the coefficients of the equation (see [11]). Fortunately, if a mask satisfies condition (0.4), then this problem in $L^2(\mathbb{R}_+)$ has a simple solution.

Proposition 6. *If a mask m of refinement equation (0.1) has the following properties:*

$$m(0) = 1, \qquad |m(\omega)|^2 + |m(\omega + 1/2)|^2 = 1 \quad \textit{for all } \omega \in [0, 1/2),$$

then the equation has a solution $\varphi \in L^2(\mathbb{R}_+)$ and moreover,

$$\|\varphi\|_{L^2(\mathbb{R}_+)} \leqslant |\widehat{\varphi}(0)|. \tag{3.5}$$

Proof. We define a function $\widehat{\varphi}(\omega)$ by equality (2.4) and prove that it belongs to $L^2(\mathbb{R}_+)$. In this case its inverse Walsh-Fourier transform φ also belongs to $L^2(\mathbb{R}_+)$ and obviously satisfies (0.1). We have

$$|\widehat{\varphi}(\omega)|^2 = \prod_{k=1}^{\infty} |m(2^{-k}\omega)|^2.$$

Since $|m(\omega)| \leqslant 1$ for all ω, it follows that for each j,

$$|\widehat{\varphi}(\omega)|^2 \leqslant \prod_{k=1}^{j} |m(2^{-k}\omega)|^2, \qquad \omega \in \mathbb{R}_+.$$

Consequently,

$$\int_0^{2^j} |\widehat{\varphi}(\omega)|^2 \, d\omega \leqslant \int_0^{2^j} \prod_{k=1}^{j} |m(2^{-k}\omega)|^2 \, d\omega = 2^j \int_0^1 \prod_{k=0}^{j} |m(2^k\omega)|^2 \, d\omega. \tag{3.6}$$

The function $|m(\omega)|^2$ is 1-periodic and piecewise constant with step 2^{-n}, therefore it is a Walsh polynomial of order 2^n-1:

$$|m(\omega)|^2=\sum_{k=0}^{2^n-1}a_k w_k(\omega).$$

The conditions imposed on the mask yield the equality $a_{2k}=\delta_{0k}/2$, therefore

$$|m(\omega)|^2=\frac{1}{2}+\sum_{s=0}^{2^{n-1}-1}a_{2s+1}w_{2s+1}(\omega).$$

Then

$$\prod_{k=0}^{j-1}|m(2^k\omega)|^2=\sum_{s_0+2s_1+\dots+2^{j-1}s_{j-1}=r}a_{s_0}\cdots a_{s_{j-1}}w_r(\omega),$$

where for each i either $s_i=0$ or s_i is odd. Hence if $r=0$, then $s_0=\dots=s_{j-1}=0$, therefore

$$\prod_{k=0}^{j-1}|m(2^k\omega)|^2=2^{-j}+\sum_{r\geqslant 1}b_r w_r(\omega).$$

Since

$$\int_0^1 w_r(\omega)\,d\omega=0$$

for all $r\geqslant 1$, it follows that

$$\int_0^1\prod_{k=0}^{j-1}|m(2^k\omega)|^2=2^{-j}$$

and we see from (3.6) that

$$\int_0^{2^j}|\widehat{\varphi}(\omega)|^2\,d\omega\leqslant 1.$$

Passing to the limit as $j\to+\infty$ we arrive at (3.5). The proof of the proposition is complete.

§ 4. The construction of dyadic wavelets in $L^2(\mathbb{R}_+)$

We recall that a function φ generates MRA in $L^2(\mathbb{R}_+)$ if the system $\{\varphi(\cdot\ominus k)\mid k\in\mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$ and the family of subspaces

$$V_j=\operatorname{clos}_{L^2(\mathbb{R}_+)}\operatorname{span}\{\varphi_{jk}\mid k\in\mathbb{Z}_+\},\qquad j\in\mathbb{Z},\tag{4.1}$$

forms MRA in $L^2(\mathbb{R}_+)$. If a compactly supported solution φ of equation (0.1) generates MRA in $L^2(\mathbb{R}_+)$, then the function

$$\psi(x)=\sum_{k=0}^{2^n-1}(-1)^k\overline{c}_{k\oplus 1}\varphi(2x\ominus k),\qquad x\in\mathbb{R}_+,$$

is a dyadic wavelet in $L^2(\mathbb{R}_+)$. Indeed, setting

$$m_1(\omega) = -w_1(\omega)\,\overline{m(\omega \oplus 1/2)}$$

we see that $\widehat{\psi}(\omega) = m_1(\omega/2)\widehat{\varphi}(\omega/2)$ and the matrix

$$\begin{pmatrix} m(\omega) & m(\omega \oplus 1/2) \\ m_1(\omega) & m_1(\omega \oplus 1/2) \end{pmatrix}$$

is unitary (see [3], Theorem 5.1.1 and [5], § 3).

We now find out when solutions of refinement equation (0.1) generate MRA in $L^2(\mathbb{R}_+)$. We start with conditions for the integer translates of the solution of equation (0.1) to form an orthonormal basis of their linear span.

Proposition 7. *Let $\varphi \in L^2(\mathbb{R}_+)$ be a compactly supported solution of refinement equation (0.1) such that $\widehat{\varphi}(0) = 1$. Then the system $\{\varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$ if and only if the mask m has no blocked sets and satisfies the relation*

$$|m(\omega)|^2 + |m(\omega + 1/2)|^2 = 1 \qquad \text{for each } \omega \in [0, 1/2). \tag{4.2}$$

Proof. We set

$$F(\omega) := \sum_{l \in \mathbb{Z}_+} |\widehat{\varphi}(\omega \oplus l)|^2. \tag{4.3}$$

The function φ has a compact support, therefore it follows by Propositions 2 and 3 that the orthonormality of the system $\{\varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ in $L^2(\mathbb{R}_+)$ is equivalent to the condition $F(\omega) \equiv 1$.

1. *Necessity*. The function φ is stable, therefore there exist no blocked sets. Collecting terms with odd and even indices in (4.3) we obtain

$$F(\omega) = |m(\omega/2)|^2 F(\omega/2) + |m(\omega/2 \oplus 1/2)|^2 F(\omega/2 \oplus 1/2). \tag{4.4}$$

Setting now $F(\omega) \equiv 1$ we arrive at (4.2).

2. *Sufficiency*. By Proposition 5 the function φ is stable. By the Strang-Fix conditions (Theorem 1) we obtain $F(0) = 1$. Let $\delta = \inf\{F(\omega) \mid \omega \in [0, 1)\}$. Since

$$\int_0^1 F(\omega)\, d\omega = 1$$

(by Proposition 6), it follows that either F is identically equal to 1 or $\delta < 1$. The function F has period 1 and is constant on dyadic intervals of range $n-1$. Therefore, either $\delta > 0$ or F vanishes on one of these intervals. The latter is impossible for in that case $\widehat{\varphi}$ possesses a periodic zero, and φ is unstable. Hence $\delta > 0$. We now set $M_\delta = \{F(\omega) = \delta \mid \omega \in [0, 1)\}$. Combining (4.2) and (4.4) we see that for each $\omega \in M_\delta$ the quantities $\omega/2$ and $\omega/2 + 1/2$ belong either to M_δ or to $\operatorname{Null} m$. This means that the set M_δ is blocked, which contradicts the assumption. Thus, $F(\omega) \equiv 1$, which completes the proof.

The following proposition is an analogue of Cohen's well-known theorem (see [13], Theorem 6.3.1).

Proposition 8. *Let*

$$m(\omega) = \frac{1}{2}\sum_{k=0}^{2^n-1} c_k w_k(\omega) \tag{4.5}$$

be a Walsh polynomial such that

$$m(0) = 1, \qquad |m(\omega)|^2 + |m(\omega + 1/2)|^2 = 1 \quad \text{for each } \omega \in [0, 1/2), \tag{4.6}$$

and let $\varphi \in L^2(\mathbb{R}_+)$ be the function defined by the formula

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(2^{-j}\omega). \tag{4.7}$$

Then the system $\{\varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$ if and only if m satisfies the modified Cohen's condition.

Proof. The function φ belongs to $L^2(\mathbb{R}_+)$ by Proposition 6. Using (4.7) we obtain the equality

$$\widehat{\varphi}(\omega) = \widehat{\varphi}(\omega/2)m(\omega/2),$$

which is equivalent to (0.1). Hence the function φ satisfies refinement equation (0.1) with mask (4.5). For each ω all the multipliers in (4.7) are equal to 1 for sufficiently large j. Indeed, the mask m is equal to 1 on $I_0^{(n)}$ and $2^{-j}\omega \to 0$ as $j \to \infty$. Hence $\widehat{\varphi}$ is W-continuous. We point out that by Proposition 1(a) the W-continuity of $\widehat{\varphi}$ also follows from Theorem 1. Thus, with the use of Proposition 3 we see that the orthonormality of the system $\{\varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ in $L^2(\mathbb{R}_+)$ is equivalent to the identity

$$\sum_{l \in \mathbb{Z}_+} |\widehat{\varphi}(\omega \oplus l)|^2 \equiv 1. \tag{4.8}$$

Assume now that (4.8) holds. Then for each $\omega \in [0, 1)$ there exists a quantity l_ω such that

$$\sum_{l=0}^{l_\omega} |\widehat{\varphi}(\omega \oplus l)|^2 > \frac{1}{2}.$$

Since $\widehat{\varphi}$ is W-continuous, it follows that for each $\omega \in [0, 1)$ there exists a dyadic interval I_ω such that

$$\sum_{l=0}^{l_\omega} |\widehat{\varphi}(t \oplus l)|^2 \geqslant \frac{1}{4}$$

for all $t \in I_\omega$. By the W-compactness of the interval $[0, 1)$ the cover $\{I_\omega \mid \omega \in [0, 1)\}$ contains a finite subcover $\{I_{\omega_1}, \ldots, I_{\omega_L}\}$. We set $l_0 = \max\{l_{\omega_1}, \ldots, l_{\omega_L}\}$. Then the inequality

$$\sum_{l=0}^{l_0} |\widehat{\varphi}(\omega \oplus l)|^2 \geqslant \frac{1}{4} \tag{4.9}$$

holds for all $\omega \in [0, 1)$.

Let $c_0 = 1/(2\sqrt{l_0+1})$. It follows from (4.9) that for each $\omega \in [0,1)$ there exists $l \in \{0,1,\ldots,l_0\}$ such that $|\widehat{\varphi}(\omega \oplus l)| \geqslant c_0$. Since $\widehat{\varphi}(0)=1$ and the function $\widehat{\varphi}$ is W-continuous, it follows that the set

$$S_0 := \bigl\{\omega \in [0,1) \bigm| |\widehat{\varphi}(\omega)| \geqslant c_0\bigr\}$$

contains a neighbourhood of zero. Consider now the following sets

$$\begin{aligned}
&S_1 := \bigl\{\omega \in [0,1) \setminus S_0 \bigm| |\widehat{\varphi}(\omega \oplus 1)| \geqslant c_0\bigr\},\\
&S_2 := \bigl\{\omega \in [0,1) \setminus (S_0 \cup S_1) \bigm| |\widehat{\varphi}(\omega \oplus 2)| \geqslant c_0\bigr\},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&S_{l_0} := \Biggl\{\omega \in [0,1) \setminus \bigcup_{l=0}^{l_0-1} S_l \Biggm| |\widehat{\varphi}(\omega \oplus l_0)| \geqslant c_0\Biggr\}.
\end{aligned}$$

The set $E = \bigcup_{l=0}^{l_0}(S_l \oplus l)$ is W-compact, congruent to $[0,1)$ modulo $\mathbb{Z}_+$, and contains a neighbourhood of zero. Since the polynomial $m(\omega)$ is equal to 1 in a neighbourhood of zero, one can choose an integer j_0 such that

$$m(2^{-j}\omega) = 1 \qquad \text{for all } j > j_0, \quad \omega \in E. \tag{4.10}$$

Combining this with (4.7) we obtain

$$|\widehat{\varphi}(\omega)| = \prod_{j=1}^{j_0} |m(2^{-j}\omega)| \cdot |\widehat{\varphi}(2^{-j_0}\omega)|, \tag{4.11}$$

where $|\widehat{\varphi}(\omega)| \geqslant c_0$ for $\omega \in E$. We observe that $|m(\omega)| \leqslant 1$ for all ω. Hence it follows by (4.10) and (4.11) that

$$|m(2^{-j}\omega)| \geqslant \prod_{l=1}^{j_0} |m(2^{-l}\omega)| \geqslant c_0 > 0 \qquad \text{for } 1 \leqslant j \leqslant j_0, \quad \omega \in E. \tag{4.12}$$

Combining now (4.10) and (4.12) we obtain

$$\inf_{j\in\mathbb{N}} \inf_{\omega\in E} |m(2^{-j}\omega)| > 0.$$

Conversely, assume that the polynomial $m(\omega)$ satisfies the modified Cohen's condition and (4.6). Then from Lemma 1 of [6] we see that the system $\{\varphi(\,\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}_+)$. This proves Proposition 8.

The following theorem gives necessary and sufficient conditions for solutions of equation (0.1) to generate MRA.

Theorem 3. *Suppose that equation* (0.1) *possesses a compactly supported* L^2*-solution* φ *such that its mask* m *satisfies conditions* (4.6) *and* $\widehat{\varphi}(0)=1$; *then the following properties are equivalent*:

(a) φ *generates MRA in* $L^2(\mathbb{R}_+)$;
(b) *the mask* m *has no blocked sets*;
(c) *the mask* m *satisfies the modified Cohen's condition.*

Proof. The implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) follow directly from Propositions 7 and 8. To establish the reverse implications we assume that m satisfies one of the conditions (b) and (c). In this case it follows by Propositions 7 and 8 that the system $\{\varphi(\,\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is orthonormal. We define $\{V_j\}$ by formula (4.1). By Proposition 4 we obtain $\bigcap V_j = \{0\}$. The embedding $V_0 \subset V_1$ follows from the fact that φ satisfies (0.1). Invoking now (4.1) we obtain the inclusion $V_j \subset V_{j+1}$ for each $j \in \mathbb{Z}$. It remains to show that

$$\overline{\bigcup V_j} = L^2(\mathbb{R}_+),$$

or, in other words,

$$\Bigl(\bigcup V_j\Bigr)^{\perp} = \{0\}. \tag{4.13}$$

Let $f \in \bigl(\bigcup V_j\bigr)^{\perp}$. For fixed positive ε we choose a dyadic entire function $u \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ such that $\|f - u\| < \varepsilon$ (in what follows we denote by $\|\cdot\|$ the norm in $L^2(\mathbb{R}_+)$). Then for each $j \in \mathbb{Z}_+$ the orthogonal projection $P_j f$ of the function f to V_j possesses the property

$$\|P_j f\|^2 = (P_j f, P_j f) = (f, P_j f) = 0.$$

Hence

$$\|P_j u\| = \|P_j(f - u)\| \leqslant \|f - u\| < \varepsilon. \tag{4.14}$$

Consider a sufficiently large integer j such that $\operatorname{supp}\widehat{u} \subset [0, 2^j)$ and $2^{-j}\omega \in [0, 2^{-n+1})$ for all $\omega \in \operatorname{supp}\widehat{u}$. Such j exists by Proposition 2(b). We set $g(\omega) = \widehat{u}(\omega)\widehat{\varphi}(2^{-j}\omega)$ and observe that the system $\{2^{-j/2}\chi(2^{-j}k, \cdot\,)\}_{k=0}^{\infty}$ is an orthonormal basis $L^2[0, 2^j]$. Hence

$$\sum_{k\in\mathbb{Z}_+} |c_k(g)|^2 = 2^{-j} \int_0^{2^j} |g(\omega)|^2 \, d\omega, \tag{4.15}$$

where

$$c_k(g) = 2^{-j} \int_0^{2^j} g(\omega)\chi(2^{-j}k, \omega)\, d\omega.$$

Taking into account the equality

$$\int_{\mathbb{R}_+} \varphi(2^j x \ominus k)\chi(x, \omega)\, dx = 2^{-j}\widehat{\varphi}(2^{-j}\omega)\chi(2^{-j}k, \omega)$$

and using Plancherel's formula we obtain

$$2^{-j/2}(u, \varphi_{jk}) = 2^{-j} \int_0^{2^j} g(\omega)\chi(2^{-j}k, \omega)\, d\omega.$$

Invoking (4.15) we see that

$$\|P_j u\|^2 = \sum_{k\in\mathbb{Z}_+} |(u, \varphi_{jk})|^2 = \int_0^{2^j} |\widehat{u}(\omega)\widehat{\varphi}(2^{-j}\omega)|^2 \, d\omega. \tag{4.16}$$

Recall that $m(\omega) = 1$ on $I_0^{(n)}$ and $2^{-j}\omega \in [0, 2^{-n+1})$ for $\omega \in \operatorname{supp}\widehat{u}$. Combining this with the equality

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(2^{-j}\omega)$$

we see that $\widehat{\varphi}(2^{-j}\omega) = 1$ for all $\omega \in \operatorname{supp}\widehat{u}$. Since $\operatorname{supp}\widehat{u} \subset [0, 2^j)$, it follows from (4.14), (4.16), and Proposition 1 that

$$\varepsilon > \|P_j u\| = \|\widehat{u}\| = \|u\|.$$

Therefore,

$$\|f\| < \varepsilon + \|u\| < 2\varepsilon\,,$$

which proves (4.13) and completes the proof of Theorem 3.

Remark 2. In the construction of classical wavelets on the real line $\mathbb{R}$ one uses the same condition (4.6) on the mask, while the condition on blocked sets is different. In the classical case the mask is a trigonometric polynomial of degree n, which has at most n zeros on the period interval. The condition in that case is as follows: the zeros of the mask cannot form a so-called non-trivial cycle ([3], Theorem 6.3.3). Another distinction, which is more important, is that the trigonometric polynomials satisfying (4.6) are found from a special Diophantine equation. For fixed n this equation has finitely many solutions of the smallest possible degree. They correspond to the Daubechies wavelets ([3], § 6.4). In the dyadic case, as seen in Theorem 3, the construction of wavelets proceeds by another scheme, which is slightly simpler. First, one chooses an arbitrary function $m(\omega)$ on $[0, 1/2)$ such that it is constant on the dyadic intervals of range n and $m(0) = 1$. Second, this function is extended onto $\mathbb{R}_+$ with the help of the equalities

$$|m(\omega)|^2 + |m(\omega + 1/2)|^2 = 1 \quad \text{and} \quad m(\omega + 1) = m(\omega).$$

Finally, one verifies (by means of a finite exhaustive search) whether m has blocked sets. If it has none, then the corresponding refinable function generates MRA in $L^2(\mathbb{R}_+)$ and a system of wavelets. According to Proposition 7 and Theorem 3, this algorithm produces all possible systems of dyadic wavelets generated by compactly supported refinable functions.

Thus, in the construction of dyadic wavelets we are free to choose an arbitrary piecewise-constant function with step 2^{-n} on the interval $[0, 1/2)$, with the unique restriction of the absence of blocked sets.

§ 5. Expansions in Walsh series and estimates of moduli of regularity

Assume that a compactly supported solution φ of equation (0.1) generates MRA in $L^2(\mathbb{R}_+)$ and is normalized so that $\widehat{\varphi}(0) = 1$. Then by Theorem 1 and Proposition 7,

$$m(0) = 1, \qquad |m(\omega)|^2 + |m(\omega + 1/2)|^2 = 1 \quad \text{for all} \quad \omega \in [0, 1/2) \tag{5.1}$$

and

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(2^{-j}\omega), \tag{5.2}$$

where m is the mask of equation (0.1). Moreover, $\operatorname{supp}\varphi \subset [0, 2^{n-1}]$. Conditions (5.1) are equivalent to the following equalities:

$$b_0 = 1, \qquad |b_l|^2 + |b_{l+2^{n-1}}|^2 = 1, \quad l \in \{0, 1, \dots, 2^{n-1} - 1\}, \tag{5.3}$$

where the b_l are the values of the mask m on the dyadic intervals $I_l^{(n)}$. If

$$l = i_1 2^0 + i_2 2^1 + \dots + i_n 2^{n-1}, \qquad i_j \in \{0, 1\},$$

then we set $c(i_1, i_2, \dots, i_n) = b_l$.

For a positive integer l we define the coefficients $a_l[m]$ by the binary expansion

$$l = \sum_{j=0}^{k} \mu_j 2^j, \qquad \mu_j \in \{0, 1\}, \qquad \mu_k = 1, \quad k = k(l), \tag{5.4}$$

as follows:

$$\begin{aligned} &a_l[m] = c(\mu_0, 0, 0, \dots, 0, 0), && \text{if } k(l) = 0, \\ &a_l[m] = c(\mu_1, 0, 0, \dots, 0, 0) c(\mu_0, \mu_1, 0, \dots, 0, 0), && \text{if } k(l) = 1, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &a_l[m] = c(\mu_k, 0, 0, \dots, 0, 0) c(\mu_{k-1}, \mu_k, 0, \dots, 0, 0) \cdots c(\mu_0, \mu_1, \mu_2, \dots, \mu_{n-2}, \mu_{n-1}), \end{aligned}$$

if $k = k(l) \geqslant n - 1$. The indices of each factor in the last product, starting with the second, are equal to the indices of the preceding factor shifted one position rightwards; at the free first position one puts the corresponding digit of the binary expansion of l.

We denote by $\mathbb{N}_0(n)$ the set of integers $l \geqslant 2^{n-1}$ with binary expansion (5.4) containing a subsequence $(\mu_j, \mu_{j+1}, \dots, \mu_{j+n-1})$ equal to $(0, 0, \dots, 0, 1)$. We observe that $a_l[m] = 0$ for $l \in \mathbb{N}_0(n)$ because

$$c(0, 0, \dots, 0, 1) = b_{2^{n-1}} = 0$$

by (5.3). Let $\mathbb{N}(n) = \mathbb{N} \setminus \mathbb{N}_0(n)$. In particular,

$$\begin{aligned} &\mathbb{N}(2) = \{2^{j+1} - 1 \mid j \in \mathbb{Z}_+\} = \{1, 3, 7, 15, 31, \dots\}, \\ &\mathbb{N}(3) = \{1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 21, 22, 23, 26, 27, 29, 30, 31, 42, \dots\}. \end{aligned}$$

Combining (5.1) and (5.2) we obtain

$$\widehat{\varphi}(\omega) = X_{1-n}(\omega) + \sum_{l \in \mathbb{N}(n)} a_l[m] X_{1-n}\left(\omega \ominus \frac{l}{2^{n-1}}\right), \tag{5.5}$$

where $X_{1-n} = \chi_{[0,1/2^{n-1})}$. Taking the inverse Walsh-Fourier transformation and using the equality

$$\int_{\mathbb{R}_+} X_{1-n}\left(\omega \ominus \frac{l}{2^{n-1}}\right) \chi(x, \omega)\, d\omega = \frac{1}{2^{n-1}} \chi_{[0,1)}\left(\frac{x}{2^{n-1}}\right) w_l\left(\frac{x}{2^{n-1}}\right),$$

we can write (5.5) in the following form:

$$\varphi(x) = \frac{1}{2^{n-1}} \chi_{[0,1)}\left(\frac{x}{2^{n-1}}\right)\left(1 + \sum_{l \in \mathbb{N}(n)} a_l[m] w_l\left(\frac{x}{2^{n-1}}\right)\right), \qquad x \in \mathbb{R}_+. \tag{5.6}$$

For $n = 2$ we obtain the Lang decomposition (0.9).

Example 4. For $n = 3$ we set

$$\begin{aligned} b_0 &= 1, \quad & b_1 &= a, \quad & b_2 &= b, \quad & b_3 &= c, \\ b_4 &= 0, \quad & b_5 &= \alpha, \quad & b_6 &= \beta, \quad & b_7 &= \gamma, \end{aligned}$$

where $|a|^2 + |\alpha|^2 = |b|^2 + |\beta|^2 = |c|^2 + |\gamma|^2 = 1$. By means of (5.6) we find a function φ with the following Walsh decomposition

$$\begin{aligned} \varphi(x) &= \frac{1}{4}\chi_{[0,1)}(y)\biggl(1 + \sum_{l\in\mathbb{N}(3)} a_l[m]w_l(y)\biggr) \\ &= \frac{1}{4}\chi_{[0,1)}(y)(1 + aw_1(y) + abw_2(y) + acw_3(y) + ab\alpha w_5(y) \\ &\qquad + ac\beta w_6(y) + ac\gamma w_7(y) + ab^2\alpha w_{10}(y) + abc\alpha w_{11}(y) + \cdots), \end{aligned} \tag{5.7}$$

where $y = x/4$. If $a = 0$ or $c = 0$, then the system $\{\varphi(\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is linearly dependent. If a and c are both non-zero, then the function φ generates MRA in $L^2(\mathbb{R}_+)$. Indeed, if $abc \neq 0$, then condition (0.11) is fulfilled for $E = [0, 1)$, and if $a \neq 0$, $b = 0$, $c \neq 0$, then it is fulfilled for $E = [0, 1/2) \cup [3/4, 1) \cup [3/2, 7/4)$. Note that in the case of $a = 1$, $0 \leqslant |b| < 1$, $c = 1$ the function φ is dyadic entire (Example 3).

The reader can find graphic illustrations related to Examples 2 and 4 in [5] and [6].

Example 5. Consider complex numbers $b_0, b_1, \dots, b_{2^{n-1}-1}$ such that

$$b_0 = 1, \qquad 0 < |b_l| \leqslant 1, \quad 1 \leqslant l \leqslant 2^{n-1} - 1,$$

and then select $b_{2^{n-1}}, b_{2^{n-1}+1}, \dots, b_{2^n-1}$ as

$$|b_{l+2^{n-1}}| = \sqrt{1 - |b_l|^2}, \qquad 0 \leqslant l \leqslant 2^{n-1} - 1.$$

Using discrete Walsh transform (0.6) we find the coefficients of the mask

$$m(\omega) = \frac{1}{2}\sum_{k=0}^{2^n-1} c_k w_k(\omega),$$

which takes the values b_l on the corresponding intervals $I_l^{(n)}$, $0 \leqslant l \leqslant 2^n - 1$. Since we choose the b_l distinct from zero for $0 \leqslant l \leqslant 2^{n-1} - 1$, it follows that $|m(\omega)| > 0$ for $\omega \in [0, 1/2)$. Consequently, the mask m satisfies the modified Cohen's condition. The corresponding refinable function φ is defined by the decomposition (5.6), where the coefficients are uniquely determined by the parameters b_l.

There exist analogues on the real line $\mathbb{R}$ of the dyadic wavelets generated by refinable functions from Example 5. These are the orthogonal Daubechies wavelets (see [3], § 6.4). In the next example, for each $n \geqslant 3$ we find explicitly a dyadic entire solution of equation (0.1).

Example 6. For arbitrary $n \geqslant 3$ we set

$$m(\omega) = \begin{cases} 1, & \text{if } \omega \in [0, 1/2 - 1/2^{n-1}) \cup [1/2 - 1/2^n, 1/2), \\ b, & \text{if } \omega \in [1/2 - 1/2^{n-1}, 1/2 - 1/2^n), \\ 0, & \text{if } \omega \in [1/2, 1/2 - 1/2^n) \cup [1 - 1/2^n, 1), \\ \beta, & \text{if } \omega \in [1 - 1/2^{n-1}, 1 - 1/2^n), \end{cases}$$

where $0 \leqslant |b| < 1$, $|\beta| = \sqrt{1 - |b|^2}$. Then it follows from (5.2) that

$$\begin{aligned} \widehat{\varphi}(\omega) = \chi_{[0,1-1/2^{n-2})}(\omega) + b\chi_{[1-1/2^{n-2},1-1/2^{n-1})}(\omega) \\ + \chi_{[1-1/2^{n-1},1)}(\omega) + \beta\chi_{[2-1/2^{n-2},2-1/2^{n-1})}(\omega) \end{aligned}$$

and therefore

$$\begin{aligned} \varphi(x) = \frac{1}{2^{n-1}}\chi_{[0,1)}\biggl(\frac{x}{2^{n-1}}\biggr)\biggl(1 + \sum_{l=1}^{2^{n-1}-3} w_l\biggl(\frac{x}{2^{n-1}}\biggr) \\ + bw_{2^{n-1}-2}\biggl(\frac{x}{2^{n-1}}\biggr) + w_{2^{n-1}-1}\biggl(\frac{x}{2^{n-1}}\biggr) + \beta w_{2^n-2}\biggl(\frac{x}{2^{n-1}}\biggr)\biggr). \end{aligned}$$

This function φ satisfies (0.11) for

$$E = \biggl[0, 1 - \frac{1}{2^{n-2}}\biggr) \cup \biggl[1 - \frac{1}{2^{n-1}}, 1\biggr) \cup \biggl[2 - \frac{1}{2^{n-2}}, 2 - \frac{1}{2^{n-1}}\biggr)$$

(if $b \neq 0$, then (0.11) holds even for $E = [0, 1)$). We point out that for $n = 3$ we arrive at the refinable function from Example 3 again.

For arbitrary $\delta > 0$ we define as follows the *dyadic modulus of continuity* of a function φ:

$$\omega(\varphi, \delta) := \sup\bigl\{|\varphi(x \oplus y) - \varphi(x)| \bigm| x, y \in [0, 2^{n-1}),\ 0 \leqslant y < \delta\bigr\}$$

(see [2], § 1.2). By [2], § 5.1, if

$$\omega(\varphi, 2^{-j}) \leqslant C2^{-\alpha j}$$

for some $\alpha > 0$, then there exists a constant $C(\varphi, \alpha)$ such that

$$\omega(\varphi, \delta) \leqslant C(\varphi, \alpha)\delta^{\alpha}. \tag{5.8}$$

We now find an estimate for the rate of decay of the sequence $\{\omega(\varphi, 2^{-j})\}$ for a fixed refinable function φ.

Let $N = 2^{n-1}$, $w_l^-(k/N) = \lim_{x\to k-0} w_l(x/N)$, and $\varphi^-(k) = \lim_{x\to k-0}\varphi(x)$. It follows from (5.6) and the equalities

$$\sum_{k=0}^{N-1} w_l\biggl(\frac{k}{N}\biggr) = \sum_{k=1}^{N} w_l^-\biggl(\frac{k}{N}\biggr) = 0, \qquad l \in \mathbb{N}(n),$$

that

$$\sum_{k=0}^{N-1}\varphi(k)=\sum_{k=1}^{N}\varphi^{-}(k)=1. \tag{5.9}$$

We define two $(N\times N)$-matrices T_0 and T_1 by their entries as follows:

$$(T_0)_{ij}=c_{2(i-1)\oplus(j-1)}\quad\text{and}\quad(T_1)_{ij}=c_{(2i-1)\oplus(j-1)}, \tag{5.10}$$

where $i,j\in\{1,2,\dots,N\}$. In particular, for $n=2$ we have

$$T_0=\begin{pmatrix}c_0 & c_1\\ c_2 & c_3\end{pmatrix},\qquad T_1=\begin{pmatrix}c_1 & c_0\\ c_3 & c_2\end{pmatrix},$$

and for $n=3$

$$T_0=\begin{pmatrix}c_0 & c_1 & c_2 & c_3\\ c_2 & c_3 & c_0 & c_1\\ c_4 & c_5 & c_6 & c_7\\ c_6 & c_7 & c_4 & c_5\end{pmatrix},\qquad T_1=\begin{pmatrix}c_1 & c_0 & c_3 & c_2\\ c_3 & c_2 & c_1 & c_0\\ c_5 & c_4 & c_7 & c_6\\ c_7 & c_6 & c_5 & c_4\end{pmatrix}.$$

For the vector-valued function $v(x):=(\varphi(x),\varphi(x+1),\dots,\varphi(x+N-1))^t$ we have

$$v(x)=\begin{cases}T_0v(2x) & \text{for } x\in[0,1/2),\\ T_1v(2x-1) & \text{for } x\in[1/2,1).\end{cases} \tag{5.11}$$

Let $e_1=(1,1,\dots,1)$ be an N-dimensional vector with all entries equal to 1. By (3.4) we obtain

$$e_1T_0=e_1T_1=e_1. \tag{5.12}$$

We denote by E_1 the orthogonal complement of the vector e_1 in $\mathbb{C}^N$:

$$E_1:=\{u=(u_1,\dots,u_N)^t\mid u_1+\dots+u_N=0\}.$$

Recall that the spectral norm of an arbitrary complex $(N\times N)$-matrix T is defined by the formula

$$\|T\|:=\sup\biggl\{\frac{\|Tu\|}{\|u\|}\Bigm| u\in\mathbb{C}^N,\ u\neq 0\biggr\},$$

where $\|u\|$ is the Euclidean norm of u. Let

$$\|T|_{E_1}\|:=\sup\biggl\{\frac{\|Tu\|}{\|u\|}\Bigm| u\in E_1,\ u\neq 0\biggr\}.$$

Similarly to Proposition 4.1 in [7] (cf. [3], § 7.2), using (5.9), (5.11), and (5.12) one can establish the following result.

Proposition 9. *Let φ be a compactly supported solution of equation (0.1) generating MRA in $L^2(\mathbb{R}_+)$ and let T_0, T_1 be $(N\times N)$-matrices with entries of the form (5.10), where $N=2^{n-1}$. Assume that for all $m\in\mathbb{N}$,*

$$\max\bigl\{\|T_{d_1}T_{d_2}\cdots T_{d_m}|_{E_1}\|\bigm| d_j\in\{0,1\},\ 1\leqslant j\leqslant m\bigr\}\leqslant Cq^m, \tag{5.13}$$

where $0<q<1$ and $C>0$. Then the function φ is W-continuous and for each integer $j\geqslant n-1$,

$$\omega(\varphi,2^{-j})\leqslant Cq^j. \tag{5.14}$$

If the function φ satisfies the assumptions of Proposition 9, then the Walsh-Fourier series (5.6) converges uniformly on $[0, 2^{n-1})$. Indeed, it follows from (5.14) that

$$\lim_{j\to\infty} j\omega(\varphi, 2^{-j}) = 0,$$

and one can apply a well-known condition of uniform convergence ([1], Theorem 2.3.5).

Obviously, for $n = 2$ the subspace E_1 is one-dimensional:

$$E_1 = \{\lambda e_2 \mid \lambda \in \mathbb{C}\}, \qquad \text{where} \quad e_2 = (1, -1)^t,$$

and for $n = 3$ the subspace E_1 has the following basis:

$$e_1^0 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad e_2^0 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \qquad e_3^0 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

This has, for example, the following consequences:

(1) if φ is as in Example 2 and $0 \leqslant |b| < 1$, then

$$\omega(\varphi, 2^{-j}) \leqslant C|b|^j, \tag{5.15}$$

in particular, for $b = 0$ the function φ is dyadic entire;

(2) if φ is as in Example 4 and $a = 1$, $0 \leqslant |\gamma| < 1$, then

$$\omega(\varphi, 2^{-j}) \leqslant C|\gamma|^j, \tag{5.16}$$

in particular, for $\gamma = 0$ the function φ is dyadic entire.

Using decompositions (0.9) and (5.7) one can easily verify that estimates (5.15) and (5.16) are asymptotically sharp (see Examples 4.3 and 4.4 in [7]). We also point out that under the assumptions of Example 3 the left hand side of inequality (5.13) vanishes (and the same holds for an arbitrary dyadic entire solution φ of (0.1)).

Remark 3. For a fixed solution φ of refinement equation (0.1) we denote by α_φ the least upper bound of the quantities $\alpha > 0$ satisfying (5.8). Then

$$\alpha_\varphi = -\log_2 \widehat{\rho}, \tag{5.17}$$

where $\widehat{\rho} = \widehat{\rho}(\mathscr{T}_0, \mathscr{T}_1)$ is the joint spectral radius of the operators $\mathscr{T}_0$ and $\mathscr{T}_1$ defined in $\mathbb{C}^N$ by their matrices T_0, T_1 and restricted to the subspace $V = \operatorname{span}\{v(x) - v(y) \mid x, y \in [0, 1)\}$ (v is the vector-valued function from (5.11)). The proof is the same as in the classical case (see [10], [11]) and we leave it out. In particular, the function φ is W-continuous if and only if $\widehat{\rho} < 1$. Formula (5.17) makes it possible to find the precise value of the Hölder exponent. However, its practical implementation is sometimes difficult because it involves the computation of the joint spectral radius. Usually, this problem can be effectively solved only for small dimensions of operators. In the above examples we have, in fact, computed the joint spectral radius $\widehat{\rho}$ for several small values of n.

Thus, for an arbitrary refinement equation it is possible, theoretically at least, to compute the exponent of regularity of the solution. Hence it is also possible to find the exponent of regularity of the corresponding wavelets. One can put the question of the magnitude of this exponent for fixed n. It is well known that for classical compactly supported wavelets the exponent of regularity is always finite and, moreover, is bounded above by the length of the support of the corresponding refinable function. The straightforward analogue of this fact does not hold for dyadic wavelets. For example, dyadic refinable functions can be entire (Examples 1 and 6). Nevertheless, it turns out that the following alternative takes place for dyadic refinable functions: either a function is dyadic entire or it has a finite regularity. The finite exponent of regularity is bounded above not by the length of the support, but by the smallest non-zero value of the modulus of the mask. We defer the precise statement to § 7. Before this, we characterize all dyadic entire refinable functions.

§ 6. Dyadic entire solutions of refinement equations

In the next result we establish conditions for the location of zeros of the mask ensuring that the solution φ of equation (0.1) is dyadic entire.

Proposition 10. *Let φ be a compactly supported L^2-solution of refinement equation (0.1) and let m be its mask. Assume that $\widehat{\varphi}(0) = 1$. Then the function φ is not dyadic entire if and only if there exists a finite sequence $\{d_k\}_{k=1}^N$, $N \leqslant 2^{n-1}$, $d_k \in \{0, 1\}$, such that*

(a) $d_1 = \dots = d_{n-1} = 0$, $d_n = 1$;

(b) *there exists an integer j, $n-1 \leqslant j \leqslant N-1$, such that $d_{j-s} = d_{N-s}$ for $s = 0, \dots, n-2$ and $d_{j-n+2} \dots d_{N-n+1} = 0$;*

(c) $m(0.d_{k+1} \dots d_{k+n}) \neq 0$ *for all* $k = 0, \dots, N-n$.

Proof. Let $\{d_k\}_{k=1}^N$ be a sequence satisfying conditions (a)–(c), and let

$$\beta = 0.d_1 \dots d_{j-n+1}(d_{j-n+2} \dots d_{N-n+1})$$

be a dyadic number with period $(d_{j-n+2} \dots d_{N-n+1})$. Since $m \in \mathscr{E}_n$ and $m(0) = 1$, it follows that for each $r \in \mathbb{N}$

$$\begin{aligned}\widehat{\varphi}(2^r\beta) = \prod_{k=1}^{\infty} m(2^{r-k}\beta) &= \prod_{k=1}^{\infty} m(d_1 \dots d_{r-k}.d_{r-k+1} \dots d_{r-k+n} \dots) \\ &= \prod_{k=1}^{\infty} m(0.d_{r-k+1} \dots d_{r-k+n}) \neq 0\end{aligned}$$

(we set $d_i = 0$ for $i \leqslant 0$, take into account the periodicity of m, and use the following property: each string of length n in the binary expansion of β coincides with some string of length n in the sequence $\{d_k\}_{k=1}^N$). Therefore, the support of the Walsh-Fourier transform $\widehat{\varphi}$ is not compact. From Proposition 2(b) we now see that the function φ is not dyadic entire.

The other way round assume that a compactly supported L^2-solution φ of equation (0.1) is not a dyadic entire function. Then $\varphi \in L^1(\mathbb{R}_+)$ and by Proposition 2, (c) the support $\operatorname{supp}\widehat{\varphi}$ is not compact. Consider an arbitrary number $\omega > 2^{2^{n-1}}$ such that $\widehat{\varphi}(\omega) \neq 0$. Its binary expansion can be written in the following form:

$$\omega = \sum_{j=0}^{l} \omega_{-j-1} 2^j + \sum_{j=1}^{\infty} \omega_j 2^{-j}, \qquad \omega_j \in \{0,1\}, \qquad \omega_{-l-1} = 1, \quad l > 2^{n-1}.$$

From (0.5) it follows that $m(2^{-s}\omega) \neq 0$ for all $s \in \mathbb{N}$. Arguing as above and using the inclusion $m \in \mathscr{E}_n$ we show that the mask m does not vanish at the following points:

$$\gamma_l = 0.0\dots 0\omega_{-l-1}, \quad \gamma_{l-1} = 0.0\dots\omega_{-l}\omega_{-l-1}, \quad \dots, \quad \gamma_0 = 0.\omega_{-n}\dots\omega_{-2}\omega_{-1}$$

Each dyadic number γ_s $(s = 0,\dots,l)$ has exactly n significant digits in its binary expansion. Since $l > 2^{n-1}$, it follows that there exist two numbers among $\gamma_0,\dots,\gamma_l$ starting with the same sequence of $n-1$ digits. Denote by r the smallest integer possessing the following property: for some $k \in \mathbb{N}$ the first $n-1$ digits of the numbers γ_r and γ_{r-k} coincide. Then the sequence $0,\dots,0,\omega_{-l-1},\dots,\omega_{-r}$ is as required and the proof of Proposition 10 is complete.

Proposition 10 reduces the question whether the solution of refinement equation (0.1) is dyadic entire to a mere exhaustive search among all binary sequences of length $2^{n-1} - n + 1$.

Corollary 6. *If a compactly supported L^2-solution φ of refinement equation* (0.1) *is dyadic entire, then*

$$\operatorname{supp}\widehat{\varphi} \subset \bigl[0, 2^{2^{n-1}}\bigr].$$

Corollary 7. *If a mask m of refinement equation* (0.1) *is equal to zero identically on the interval $[2^{-r}, 2^{-r+1})$ for some $r \in \mathbb{N}$, then the solution φ is dyadic entire.*

Proof. Since the mask m is equal to 1 on all the intervals $[k, k+2^{-n})$, $k \in \mathbb{Z}_+$, it follows that $1 \leqslant r \leqslant n$. Hence $m(\omega) = 0$ whenever the binary expansion of ω starts with the sequence $0.0\dots01$ (r consecutive zeros followed by one). Therefore, there exists for the mask no sequence $\{d_k\}_{k=1}^N$ satisfying conditions (a)–(c) from Proposition 10.

In particular, if a mask is equal to zero on the interval $[1/2, 1)$, then the solution φ of equation (0.1) is dyadic entire. We also point out that if

$$m(\omega) = \begin{cases} 1 & \text{for } \omega \in [0, 2^{-r}), \\ 0 & \text{for } \omega \in [2^{-r}, 2^{-r+1}), \end{cases}$$

where $r \in \mathbb{N}$, $r \leqslant n$, then $\varphi(x) = 2^{1-r}\chi_{[0,2^{r-1})}(x)$ regardless of the values of the mask at all other points.

§ 7. An alternative for smooth refinable functions

Thus, for an arbitrary refinement equation one can decide by means of a finite exhaustive search whether the solution is dyadic entire. It turns out that if a solution φ of equation (0.1) is not dyadic entire, then it has a finite regularity.

Moreover, the Hölder exponent α_φ has an effective upper estimate by the smallest non-zero value of the modulus of the mask, and this upper bound is sharp.

The modulus of the mask $|m(\omega)|$ is a piecewise constant function with step 2^{-n}, therefore it can take 2^n values at most. One of these values is zero because by the modified Strang-Fix condition (Theorem 1) the mask must vanish on the period interval. Let a_m be the smallest non-zero value of the modulus of the mask m:

$$a_m = \min\bigl\{|m(\omega)| \bigm| \omega \in \mathbb{R}_+,\ m(\omega) \neq 0\bigr\}.$$

Since the mask has period 1 and is constant on the dyadic intervals of range n, the value of a_m coincides with the smallest non-zero number in the set $\{|m(2^{-n}k)|, k = 0, \dots, 2^{n-1}\}$.

Theorem 4. *The following alternative holds for each compactly supported L^2-solution φ of equation (0.1) satisfying $\widehat{\varphi}(0) = 1$:*

(1) *φ is a dyadic entire function;*
(2) *$a_m < 1$ and $\alpha_\varphi \leqslant -\log_2 a_m$.*

Proof. Assume that φ is not a dyadic entire function; then by Proposition 2(c) the support of the function $\widehat{\varphi}$ is not compact. This means that there exists arbitrarily large ω for which $\widehat{\varphi}(\omega) \neq 0$. If we define a positive integer k by the inequalities $2^k \leqslant \omega < 2^{k+1}$ and take into account the equalities $m(2^{-j}\omega) = 1$ for all $j \geqslant n+k+1$, then we obtain

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(2^{-j}\omega) = \prod_{j=1}^{n+k} m(2^{-j}\omega). \tag{7.1}$$

Since $\widehat{\varphi}(\omega) \neq 0$, for each $j = 1, \dots, n+k$ the quantity $|m(2^{-j}\omega)|$ is non-zero and is therefore at least a_m. Substituting the inequality $|m(2^{-j}\omega)| \geqslant a_m$ in (7.1) we obtain

$$|\widehat{\varphi}(\omega)| \geqslant (a_m)^{n+k}. \tag{7.2}$$

If we now assume that $a_m \geqslant 1$, then there exists by (7.2) arbitrarily large ω such that $|\widehat{\varphi}(\omega)|^2 \geqslant 1$. However, since the function $|\widehat{\varphi}(\omega)|^2$ is constant on the dyadic intervals of range $n-1$, it is not integrable on the half-line $\mathbb{R}_+$. This means that $\widehat{\varphi} \notin L^2(\mathbb{R}_+)$, and therefore $\varphi \notin L^2(\mathbb{R}_+)$, which contradicts the assumption.

Thus, $a_m < 1$. Substituting the inequality $k \leqslant \log_2 \omega$ in (7.2) we obtain the inequality $|\widehat{\varphi}(\omega)| \geqslant (a_m)^n / \omega^{-\log_2 a_m}$. On the other hand $|\widehat{\varphi}(\omega)| \leqslant (1/2)\omega(\varphi, 2/\omega)$ for each $\omega > 0$ (see [2], Ch. 9, exercise 9.12). Hence there exists a positive constant C such that $|\widehat{\varphi}(\omega)| \leqslant C/\omega^{\alpha_\varphi}$ for all $\omega \geqslant 1$. Therefore, $\alpha_\varphi \leqslant -\log_2 a_m$ and the proof of Theorem 4 is complete.

Remark 4. There exist infinitely smooth functions in $\mathbb{R}_+$ that are not dyadic entire. For refinable functions, however, this situation is impossible by Theorem 4. The estimate $\alpha_\varphi \leqslant -\log_2 a_m$ is sharp for refinable functions. It is attained, for instance, on the Lang functions (Example 2). By contrast with the situation on the real line $\mathbb{R}$, where for each n there exists a unique refinable function of maximum smoothness (the cardinal B-spline of order $n-1$; see, for instance, [12], [13]), the regularity of solutions of equation (0.1) is unbounded for fixed n. However, it is bounded for functions that are not dyadic entire for fixed value of a_m. This bound is always attained for fixed n and a_m on compactly supported dyadic refinable functions.

Thus, for dyadic wavelets we can verify in finite time whether a wavelet function is dyadic entire. If not, then we can find an upper estimate of its exponent of regularity. Another useful consequence of Theorem 4 concerns the case when the modulus of the mask takes only two values: 0 and 1, in particular, if the mask is the indicator function of several dyadic intervals of range n.

Corollary 8. *If the modulus of the mask of equation* (0.1) *takes values* 0 *and* 1, *then the* L^2*-solution of this equation is dyadic entire.*

Proposition 10 provides an algorithm testing whether refinable functions belong to the class of dyadic entire functions. For masks with modulus taking only two values this algorithm actually gives one a criterion for the solubility of the refinement equations in $L^2(\mathbb{R}_+)$ (Corollary 8).

Using Theorem 3 we obtain the following result.

Corollary 9. *If a function* φ *generates MRA in* $L^2(\mathbb{R}_+)$, *and the modulus of its mask takes values* 0 *and* 1, *then this function and the corresponding wavelets are dyadic entire.*

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