Biorthogonal dyadic wavelets on $\mathbb{R}_+$

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A compactly supported function $\varphi \in L^2(\mathbb{R}_+)$ is called a *scaling function* if it satisfies an equation of the type

$$\varphi(x) = \sum_{k=0}^{2^n-1} c_k \varphi(2x \ominus k), \qquad x \in \mathbb{R}_+, \tag{1}$$

where the c_k are complex coefficients and $\ominus$ is the dyadic subtraction on the positive half-line $\mathbb{R}_+$ (about this operation and some other concepts used below see [1], [2]). Using the Walsh–Fourier transform, we get from (1) that $\widehat{\varphi}(\omega) = m(\omega/2)\widehat{\varphi}(\omega/2)$, where $m(\omega) = (1/2)\sum_{k=0}^{2^n-1} c_k w_k(\omega)$ is a Walsh polynomial called the *mask* of φ. A subset M of $[0,1)$ is said to be a *blocked set* for the mask m if it is a union of dyadic intervals of rank $n-1$, does not contain the interval $[0, 2^{-n+1})$, and is such that each point in $(M/2) \cup (1/2 + M/2)$ and not in M is a zero of m. The following is proved in [2].

Proposition 1. *If a scaling function φ satisfies the equation* (1) *and $\widehat{\varphi}(0) = 1$, then* $\operatorname{supp}\varphi \subset [0, 2^{n-1}]$ *and $\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(2^{-j}\omega)$. Moreover, the system $\{\varphi(\,\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is linearly independent if and only if the mask m has no blocked sets.*

Let φ and $\widetilde{\varphi}$ be two scaling functions in $L^2(\mathbb{R}_+)$ with masks

$$m(\omega) = \frac{1}{2}\sum_{k=0}^{2^n-1} c_k w_k(\omega), \qquad \widetilde{m}(\omega) = \frac{1}{2}\sum_{k=0}^{2^{\tilde{n}}-1} \widetilde{c}_k w_k(\omega),$$

respectively, such that $\widehat{\varphi}(0) = \widehat{\widetilde{\varphi}}(0) = 1$. We consider the following systems of integral translates of φ and $\widetilde{\varphi}$:

$$\{\varphi(\,\cdot \ominus k) \mid k \in \mathbb{Z}_+\}, \qquad \{\widetilde{\varphi}(\,\cdot \ominus k) \mid k \in \mathbb{Z}_+\}. \tag{2}$$

The polynomial $m^*(\omega) = m(\omega)\,\overline{\widetilde{m}(\omega)}$ is the mask of the scaling function $\varphi^*(x) = \displaystyle\int_{\mathbb{R}_+} \varphi(t \ominus x)\,\overline{\widetilde{\varphi}(t)}\,dt$. As in Proposition 2.5.2 in [3], if the systems (2) are biorthonormal in $L^2(\mathbb{R}_+)$ then

$$m^*(\omega) + m^*\left(\omega + \frac{1}{2}\right) = 1 \quad \text{for all } \omega \in \mathbb{R}_+. \tag{3}$$

By Proposition 1, this implies that if one of the masks m, $\widetilde{m}$, m^* has a blocked set, then the systems (2) are not biorthonormal in $L^2(\mathbb{R}_+)$.

Theorem 1 (cf. [3], Theorem 2.5.6). *Let m^* satisfy the condition* (3)*. Then the systems* (2) *are biorthonormal in $L^2(\mathbb{R}_+)$ if and only if there exists a W-compact set E congruent to $[0,1)$ modulo $\mathbb{Z}_+$, containing a neighbourhood of zero, and such that*

$$\inf_{j\in\mathbb{N}} \inf_{\omega\in E} |m(2^{-j}\omega)| > 0, \qquad \inf_{j\in\mathbb{N}} \inf_{\omega\in E} |\widetilde{m}(2^{-j}\omega)| > 0. \tag{4}$$

Note that if $m^*(\omega) \neq 0$ for all $\omega \in [0, 1/2)$, then the inequalities (4) hold for $E = [0,1)$.

A *multiresolution analysis* (*MRA*) in $L^2(\mathbb{R}_+)$ is a sequence of closed subspaces $V_j \subset L^2(\mathbb{R}_+)$, $j \in \mathbb{Z}$, satisfying the following conditions: (i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

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(ii) the union $\bigcup V_j$ is dense in $L^2(\mathbb{R}_+)$ and $\bigcap V_j = \{0\}$; (iii) $f(\cdot) \in V_j \iff f(2\,\cdot\,) \in V_{j+1}$ for all $j \in \mathbb{Z}$; (iv) $f(\cdot) \in V_0 \implies f(\,\cdot \oplus k) \in V_0$ for all $k \in \mathbb{Z}_+$; (v) there exists a function $\varphi \in L^2(\mathbb{R}_+)$ such that the system $\{\varphi(\,\cdot \ominus k) \mid k \in \mathbb{Z}_+\}$ is a Riesz basis in V_0.

For any function $f \in L^2(\mathbb{R}_+)$, let $f_{j,k}(x) = 2^{j/2} f(2^j x \ominus k)$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}_+$. Furthermore, we say that a *function* $\varphi \in L^2(\mathbb{R}_+)$ *generates an MRA in* $L^2(\mathbb{R}_+)$ if the functions $\varphi(\,\cdot \ominus k)$, $k \in \mathbb{Z}_+$, form a Riesz system in $L^2(\mathbb{R}_+)$ and, in addition, the family of subspaces $V_j = \overline{\operatorname{span}\{\varphi_{j,k} \mid k \in \mathbb{Z}_+\}}$, $j \in \mathbb{Z}$, is an MRA in $L^2(\mathbb{R}_+)$.

Given two MRAs $\{V_j\}$ and $\{\widetilde{V}_j\}$ in $L^2(\mathbb{R}_+)$, we say that two functions $\psi \in V_1$ and $\widetilde{\psi} \in \widetilde{V}_1$ form a *biorthogonal wavelet pair* if $\psi \perp \widetilde{V}_0$, $\widetilde{\psi} \perp V_0$, and $\bigl(\psi(\,\cdot \oplus k), \widetilde{\psi}(\,\cdot \oplus l)\bigr) = \delta_{k,l}$, $k, l \in \mathbb{Z}_+$. As usual, $\mathscr{M}^*$ denotes the matrix conjugate to $\mathscr{M}$ and I is the identity matrix.

Proposition 2. *Let $\{V_j\}$ and $\{\widetilde{V}_j\}$ be two MRAs generated by scaling functions φ and $\widetilde{\varphi}$, respectively, and suppose that the systems* (2) *are biorthonormal. If the matrices*

$$\mathscr{M} = \begin{pmatrix} m(\omega) & m(\omega \oplus 1/2) \\ m_1(\omega) & m_1(\omega \oplus 1/2) \end{pmatrix}, \qquad \widetilde{\mathscr{M}} = \begin{pmatrix} \widetilde{m}(\omega) & \widetilde{m}(\omega \oplus 1/2) \\ \widetilde{m}_1(\omega) & \widetilde{m}_1(\omega \oplus 1/2) \end{pmatrix}$$

satisfy the condition $\mathscr{M}\widetilde{\mathscr{M}}^* = I$ for almost all $\omega \in [0,1)$, then ψ and $\widetilde{\psi}$ given by the equalities

$$\widehat{\psi}(\omega) = m_1(\omega/2)\,\widehat{\varphi}(\omega/2), \qquad \widehat{\widetilde{\psi}}(\omega) = \widetilde{m}_1(\omega/2)\,\widehat{\widetilde{\varphi}}(\omega/2) \tag{5}$$

form a biorthogonal wavelet pair. In particular, we can choose

$$m_1(\omega) = -w_1(\omega)\,\overline{m(\omega \oplus 1/2)}, \qquad \widetilde{m}_1(\omega) = -w_1(\omega)\,\overline{\widetilde{m}(\omega \oplus 1/2)}. \tag{6}$$

Theorem 2 (cf. [3], Theorem 2.7.5). *Let $\{V_j\}$ and $\{\widetilde{V}_j\}$ be two MRAs generated by scaling functions φ and $\widetilde{\varphi}$, respectively, whose masks satisfy the condition* (3) *and the conditions $m(1/2) = \widetilde{m}(1/2) = 0$, and let ψ and $\widetilde{\psi}$ be defined by* (5) *and* (6)*. Then each of the systems $\{\psi_{j,k}\}$ and $\{\widetilde{\psi}_{j,k}\}$ is a frame in $L^2(\mathbb{R}_+)$. Moreover, if the systems* (2) *are biorthonormal, then ψ and $\widetilde{\psi}$ form a biorthogonal wavelet pair and each of the systems $\{\psi_{j,k}\}$ and $\{\widetilde{\psi}_{j,k}\}$ is a Riesz basis in $L^2(\mathbb{R}_+)$.*

Similar results can be proved for biorthogonal wavelet systems on the Cantor group and the Vilenkin groups (the orthogonal case was studied in [4], [5]).

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