

## Multiresolution Analysis and Wavelets on Vilenkin Groups

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**Abstract:** This paper gives a review of multiresolution analysis and compactly supported orthogonal wavelets on Vilenkin groups. The Strang-Fix condition, the partition of unity property, the linear independence, the stability, and the orthonormality of "integer shifts" of the corresponding refinable functions are considered. Necessary and sufficient conditions are given for refinable functions to generate a multiresolution analysis in the  $L^2$ -spaces on Vilenkin groups. Several examples are provided to illustrate these results.

**Keywords:** . Walsh transform, lacunary Walsh series, Cantor group, Vilenkin group, Vilenkin-Chrestenson transform, wavelets, stability.

### 1 Introduction

IT is well-known that the Walsh system is the group of characters of the Cantor group (the dyadic or 2-series local field). It was discovered independently by Fine [1] and Vilenkin [2]. The latter actually introduced a large class of locally compact abelian groups (now called Vilenkin groups) and which includes the Cantor group as a special case. The books [3–6] are the main references to harmonic analysis on these groups. See also [7] for applications of the Cantor group to the theory of lacunary trigonometric series. Orthogonal compactly supported wavelets on the Cantor group (and relevant wavelets on the positive half-line  $\mathbb{R}_+$ ) are studied in [8–11]. Decimation by an integer  $p$  different from 2 is discussed in [12–14], but construction for a general  $p$  is not completely treated. Here we review some of the elements of that construction and give an approach to the  $p = 3$  case in a concrete fashion.

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We define Vilenkin's group  $G$  as the group of sequences

$$x = (x_j) = (\dots, 0, 0, x_k, x_{k+1}, x_{k+2}, \dots),$$

where  $x_j \in \{0, 1, \dots, p-1\}$  for  $j \in \mathbb{Z}$  and  $x_j = 0$  for  $j < k = k(x)$ . The group operation on  $G$  is denoted by  $\oplus$  and is defined as coordinatewise addition modulo  $p$ :

$$(z_j) = (x_j) \oplus (y_j) \iff z_j = x_j + y_j \pmod{p} \quad \text{for } j \in \mathbb{Z},$$

and topology in  $G$  is introduced via the complete system of neighbourhoods of zero

$$U_l = \{(x_j) \in G \mid x_j = 0 \text{ for } j \leq l\}, \quad l \in \mathbb{Z},$$

(e.g., [3]). Put  $U = U_0$  and denote by  $\ominus$  the inverse operation of  $\oplus$  (so, if  $\theta$  is the zero sequence, then  $x \ominus x = \theta$ ).

The Lebesgue spaces  $L^q(G)$ ,  $1 \leq q \leq \infty$ , are defined by the Haar measure  $\mu$  on Borel's subsets of  $G$  normalized by  $\mu(U) = 1$  (see, e.g., [3]). Denote by  $(\cdot, \cdot)$  and  $\|\cdot\|$  the inner product and the norm in  $L^2(G)$  respectively.

The group dual to  $G$  is denoted by  $G^*$  and consists of all sequences of the form

$$\omega = (\omega_j) = (\dots, 0, 0, \omega_k, \omega_{k+1}, \omega_{k+2}, \dots),$$

where  $\omega_j \in \{0, 1, \dots, p-1\}$  for  $j \in \mathbb{Z}$  and  $\omega_j = 0$  for  $j < k = k(\omega)$ . The operations of addition and subtraction, the neighbourhoods  $\{U_l^*\}$  and the Haar measure  $\mu^*$  for  $G^*$  are introduced as above for  $G$ . Each character on  $G$  can be defined by the formula

$$\chi(x, \omega) = \exp\left(\frac{2\pi i}{p} \sum_{j \in \mathbb{Z}} x_{-j} \omega_{j-1}\right), \quad x \in G,$$

for some  $\omega \in G^*$  (see, e.g., [5]).

Take in  $G$  a discrete subgroup  $H = \{(x_j) \in G \mid x_j = 0 \text{ for } j > 0\}$  and define an automorphism  $A \in \text{Aut } G$  by the formula  $(Ax)_j = x_{j+1}$ . It is easy to see that the quotient group  $H/A(H)$  contains  $p$  elements and the annihilator  $H^\perp$  of the subgroup  $H$  consists of all sequences  $(\omega_j) \in G^*$  which satisfy  $\omega_j = 0$  for  $j > 0$ .

We define a map  $\lambda : G \rightarrow \mathbb{R}_+$  by

$$\lambda(x) = \sum_{j \in \mathbb{Z}} x_j p^{-j}, \quad x = (x_j) \in G.$$

The image of  $H$  under  $\lambda$  is the set of non-negative integers:  $\lambda(H) = \mathbb{Z}_+$ . For every  $\alpha \in \mathbb{Z}_+$ , let  $h_{[\alpha]}$  denote the element of  $H$  such that  $\lambda(h_{[\alpha]}) = \alpha$ . For  $G^*$ , we define the map  $\lambda^* : G^* \rightarrow \mathbb{R}_+$ , the automorphism  $B \in \text{Aut } G^*$ , the subgroup  $U^*$  and

the elements  $\omega_{[\alpha]}$  of  $H^\perp$  similarly to  $\lambda, A, U$  and  $h_{[\alpha]}$  respectively. We note that  $\chi(Ax, \omega) = \chi(x, B\omega)$  for all  $x \in G, \omega \in G^*$ .

The *generalized Walsh functions* for  $G$  can be defined by

$$W_\alpha(x) = \chi(x, \omega_{[\alpha]}), \quad \alpha \in \mathbb{Z}_+, x \in G.$$

These functions are continuous on  $G$  and satisfy the orthogonality relations

$$\int_U W_\alpha(x) \overline{W_\beta(x)} d\mu(x) = \delta_{\alpha,\beta}, \quad \alpha, \beta \in \mathbb{Z}_+,$$

where  $\delta_{\alpha,\beta}$  is the Kronecker delta. It is well-known that the system  $\{W_\alpha\}$  is complete in  $L^2(U)$ . The corresponding system for  $G^*$  is defined by

$$W_\alpha^*(\omega) = \chi(h_{[\alpha]}, \omega), \quad \alpha \in \mathbb{Z}_+, \omega \in G^*.$$

The system  $\{W_\alpha^*\}$  is an orthonormal basis of  $L^2(U^*)$ .

For any positive interger  $n$  let  $\mathcal{E}_n(G)$  denotes the collection of all functions on  $G$  which are constant on

$$U_{n,\alpha} = A^{-n}(h_{[\alpha]}) \oplus A^{-n}(U)$$

for each  $\alpha \in \mathbb{Z}_+$ . The class  $\mathcal{E}_n(G^*)$  is defined in a similar way.

As usual, we denote by  $\widehat{f}$  the Fourier transform of  $f$ . According to Proposition 2 in [14] (see also [5]).the following properties hold:

- (a) if  $f \in L^1(G) \cap \mathcal{E}_n(G)$ , then  $\text{supp } \widehat{f} \subset U_{-n}^*$ ;
- (b) if  $f \in L^1(G)$  and  $\text{supp } f \subset U_{-n}$ , then  $\widehat{f} \in \mathcal{E}_n(G^*)$ .

In the sequel,  $\mathbf{1}_E$  stands for the characteristic function of a subset  $E$  of  $G$ .

## 2 Stability of Refinable Functions

Let  $L_c^2(G)$  be the set of all compactly supported functions in  $L^2(G)$ . We say that a function  $\varphi \in L_c^2(G)$  is a *refinable function*, if it satisfies an equation of the type

$$\varphi(x) = p \sum_{\alpha=0}^{p^n-1} a_\alpha \varphi(Ax \ominus h_{[\alpha]}). \quad (1)$$

The functional equation (1) is called the *refinement equation*. The generalized Walsh polynomial

$$m(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W_\alpha^*(\omega)} \quad (2)$$

is called the *mask* of equation (1) (or the mask of its solution  $\varphi$ ).

EXAMPLE 1. If  $a_0 = \dots = a_{p-1} = 1/p$  and  $a_\alpha = 0$  for all  $\alpha \geq p$ , then a solution of equation (1) is  $\varphi = \mathbf{1}_{U_{n-1}}$ ; in particular, the Haar function:  $\varphi = \mathbf{1}_U$  satisfies this equation when  $n = 1$  (compare with [12], Remark 1.3, [15]).

The sets

$$U_{n,s}^* := B^{-n}(\omega_{[s]}) \oplus B^{-n}(U^*), \quad 0 \leq s \leq p^n - 1,$$

are cosets of the subgroup  $B^{-n}(U^*)$  in the group  $U^*$ . For every  $0 \leq \alpha \leq p^n - 1$  the Walsh function  $W_\alpha^*(\cdot)$  is constant on each  $U_{n,s}^*$ . Thus, the mask  $m$  belongs to  $\mathcal{E}_n(G^*)$ .

It was noted in [12] that the coefficients of equation (1) are related to the values  $b_s$  of  $m$  on cosets  $U_{n,s}^*$  by means of the direct and the inverse Vilenkin-Chrestenson transforms:

$$a_\alpha = \frac{1}{p^n} \sum_{s=0}^{p^n-1} b_s W_\alpha^*(B^{-n}\omega_{[s]}), \quad 0 \leq \alpha \leq p^n - 1, \quad (3)$$

$$b_s = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W_\alpha^*(B^{-n}\omega_{[s]})}, \quad 0 \leq s \leq p^n - 1. \quad (4)$$

They can be realized by the fast algorithms (see, for instance, [6] p.463, [16]). Thus, any choice of the values of  $m$  on  $U_{n,s}^*$  defines also the coefficients of equation (1).

THEOREM 1. Let  $\varphi \in L_c^2(G)$  be a solution of the refinement equation (1), and let  $\widehat{\varphi}(\theta) = 1$ . Then

$$\sum_{\alpha=0}^{p^n-1} a_\alpha = 1, \quad \text{supp } \varphi \subset U_{1-n},$$

and

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m(B^{-j}\omega).$$

Moreover, the following properties are true: :

1.  $\widehat{\varphi}(h^*) = 0$  for all  $h^* \in H^\perp \setminus \{\theta\}$  (the modified Strang-Fix condition);
2.  $\sum_{h \in H} \varphi(x \oplus h) = 1$  for almost every  $x \in G$  (the partition of unity property).

A function  $f \in L^2(G)$  is said to be *stable* if there exist positive constants  $A_0$  and  $B_0$  such that

$$A_0 \left( \sum_{\alpha=0}^{\infty} |a_\alpha|^2 \right)^{1/2} \leq \left\| \sum_{\alpha=0}^{\infty} a_\alpha f(\cdot \ominus h_{[\alpha]}) \right\| \leq B_0 \left( \sum_{\alpha=0}^{\infty} |a_\alpha|^2 \right)^{1/2}$$

for each sequence  $\{a_\alpha\} \in \ell^2$ . In other words, a function  $f$  is stable in  $L^2(G)$  if functions  $f(\cdot \ominus h)$ ,  $h \in H$ , form a Riesz system in  $L^2(G)$ . Note also, that a function  $f$  is stable in  $L^2(\mathbb{R}_+)$  with constants  $A_0$  and  $B_0$  if and only if

$$A_0 \leq \sum_{h^* \in H^\perp} |\widehat{f}(\omega \ominus h^*)|^2 \leq B_0 \quad \text{for a.e. } \omega \in G^* \quad (5)$$

(the proof of this fact is quite similar to that of Theorem 1.1.7 in [17]). We say that a function  $g : G^* \rightarrow \mathbb{C}$  has a *periodic zero* at a point  $\omega \in G^*$  if  $g(\omega \oplus h^*) = 0$  for all  $h^* \in H^\perp$ .

**THEOREM 2.** *For any  $f \in L_c^2(G)$  the following properties are equivalent :*

- (a)  $f$  is stable in  $L^2(G)$ ;
- (b)  $\{f(\cdot \ominus h) | h \in H\}$  is a linearly independent system;
- (c) the Fourier transform of  $f$  does not have periodic zeros.

**PROOF.** The implication (a)  $\Rightarrow$  (b) follows from the well-known property of the Riesz systems (see, e.g., [17], Theorem 1.1.2). Our next claim is that  $f \in L^1(G)$ , since  $f$  has compact support and  $f \in L^2(G)$ . Let us choose a positive integer  $n$  such that  $\text{supp } f \subset U_{1-n}$ . As noted in Introduction, then  $\widehat{f} \in \mathcal{E}_{n-1}(G^*)$ . Besides, if  $\lambda(h) > p^{n-1}$ ,

$$\mu\{\text{supp } f(\cdot \ominus h) \cap U_{1-n}\} = 0.$$

Therefore, the linearly independence of the system  $\{f(\cdot \ominus h) | h \in H\}$  is equivalent to that of the finite system  $\{f(\cdot \ominus h_{[\alpha]}) | \alpha = 0, 1, \dots, p^{n-1} - 1\}$ . Further, if some vector  $(a_0, \dots, a_{p^{n-1}-1})$  satisfies the conditions

$$\sum_{\alpha=0}^{p^{n-1}-1} a_\alpha f(\cdot \ominus \alpha) = 0 \quad \text{and} \quad |a_0| + \dots + |a_{p^{n-1}-1}| > 0, \quad (6)$$

then using the Fourier transform we obtain

$$\widehat{f}(\omega) \sum_{\alpha=0}^{p^{n-1}-1} a_\alpha W_\alpha^*(\omega) = 0.$$

The Walsh polynomial

$$W^*(\omega) = \sum_{\alpha=0}^{p^{n-1}-1} a_\alpha W_\alpha^*(\omega)$$

is not identically equal to zero; hence, among  $U_{n-1,s}^*$ ,  $0 \leq s \leq p^{n-1} - 1$ , there exists a set (denote it by  $X$ ) for which  $W^*(X \oplus h^*) \neq 0$ ,  $h^* \in H^\perp$ . Since  $\widehat{f} \in \mathcal{E}_{n-1}(G^*)$ , it follows that (6) holds if and only if there exists a set  $X = U_{n-1,s}^*$ ,  $X \subset U^*$ , such that  $\widehat{f}(X \oplus h^*) = 0$  for all  $h^* \in H^\perp$ . Thus, (b)  $\Leftrightarrow$  (c).

It remains to prove that (c)  $\Rightarrow$  (a). Suppose that  $\widehat{f}$  does not have periodic zeros. Then

$$F(\omega) := \sum_{h^* \in H^\perp} |\widehat{f}(\omega \ominus h^*)|^2$$

is positive and  $H^\perp$ -periodic function. Moreover, since  $\widehat{f} \in \mathcal{E}_{n-1}(G^*)$ , we see that  $F$  is constant on each  $U_{n-1,s}^*$ ,  $0 \leq s \leq p^{n-1} - 1$ . Therefore (5) is satisfied and so Theorem 2 is established (note that in [14] this theorem was proved in a different way).

Let  $M \subset U^*$  and let

$$T_p M = \bigcup_{l=0}^{p-1} \left\{ B^{-1} \omega_{[l]} + B^{-1}(\omega) \mid \omega \in M \right\}.$$

The set  $M$  is said to be *blocked* (for the mask  $m$ ) if it coincides with some union of the sets  $U_{n-1,s}^*$ ,  $0 \leq s \leq p^{n-1} - 1$ , does not contain the set  $U_{n-1,0}^*$ , and satisfies the condition

$$T_p M \subset M \cup \{\omega \in U^* \mid m(\omega) = 0\}.$$

The notion of a blocked set was introduced by the author and V. Protasov in [11] in the setting of dyadic wavelets on  $\mathbb{R}_+$ . With the help of Theorem 2 can be proved the following

**THEOREM 3.** *Let  $\varphi \in L_c^2(G)$  be a refinable function in  $L^2(G)$  such that  $\widehat{\varphi}(\theta) = 1$ . Then  $\varphi$  is not stable if and only if its mask  $m$  possesses a blocked set.*

It is clear that each mask can have only a finite number of blocked sets. Thus, Theorem 3 reduces the stability problem for a refinable function to the verification of some combinatorial property, which can be verified, at least theoretically, in finite time.

### 3 Multiresolution Analysis on Vilenkin's Group

A collection of closed subspaces  $V_j \subset L^2(G)$ ,  $j \in \mathbb{Z}$  is called a *multiresolution analysis (MRA)* in  $L^2(G)$  if the following hold:

- (i)  $V_j \subset V_{j+1}$  for all  $j \in \mathbb{Z}$ ;
- (ii)  $\overline{\bigcup V_j} = L^2(G)$  and  $\bigcap V_j = \{0\}$ ;
- (iii)  $f(\cdot) \in V_j \iff f(A\cdot) \in V_{j+1}$  for all  $j \in \mathbb{Z}$ ;
- (iv)  $f(\cdot) \in V_0 \implies f(\cdot \ominus h) \in V_0$  for all  $h \in H$ ;
- (v) there is a function  $\varphi \in L^2(G)$  such that the system  $\{\varphi(\cdot \ominus h) \mid h \in H\}$  is an orthonormal basis of  $V_0$ .

The function  $\varphi$  in condition (v) is called a *scaling function* in  $L^2(G)$ .

For arbitrary  $\varphi \in L^2(G)$  we set

$$\varphi_{j,h}(x) = p^{j/2} \varphi(A^j x \ominus h), \quad j \in \mathbb{Z}, h \in H.$$

We say that a function  $\varphi$  generates a MRA in  $L^2(G)$  if the system  $\{\varphi(\cdot \ominus h) \mid h \in H\}$  is orthonormal in  $L^2(G)$  and, in addition, the family of subspaces

$$V_j = \text{clos}_{L^2(G)} \text{span} \{\varphi_{j,h} \mid h \in H\}, \quad j \in \mathbb{Z},$$

is a MRA in  $L^2(G)$ . If a function  $\varphi$  generates a MRA in  $L^2(G)$ , then it is a scaling function in  $L^2(G)$ . In this case the system  $\{\varphi_{j,h} \mid h \in H\}$  is an orthonormal basis of  $V_j$  for every  $j \in \mathbb{Z}$  and one can define *orthogonal wavelets*  $\psi_1, \dots, \psi_{p-1}$  in such a way that the functions

$$\psi_{l,j,h}(x) = p^{j/2} \psi_l(A^j x \ominus h), \quad 1 \leq l \leq p-1, j \in \mathbb{Z}, h \in H,$$

form an orthonormal basis of  $L^2(G)$  (see Section 5). Note that in Example 1 we can take

$$\psi_l(x) = \sum_{\alpha=0}^{p-1} \varepsilon_p^{l\alpha} \varphi(Ax \ominus h_{[\alpha]}), \quad 1 \leq l \leq p-1,$$

where  $\varepsilon_p = \exp(2\pi i/p)$ .

Let us denote by  $\delta_l$  the sequence  $\omega = (\omega_j)$  such that  $\omega_1 = l$  and  $\omega_j = 0$  for  $j \neq 1$  (in particular,  $\delta_0 = \theta$ ). It is easily seen that

$$\{\omega \in H^* \mid \chi(x, \omega) = 1 \text{ for } x \in A(H)\} = \{\delta_0, \delta_1, \dots, \delta_{p-1}\}.$$

Hence the set  $\{\delta_l\}$  is the annihilator of the subgroup  $A(H)$  in  $H$ . It was claimed in [12] that if a refinable function  $\varphi$  satisfies the condition  $\widehat{\varphi}(\theta) = 1$  and the orthonormality of  $\{\varphi(\cdot \ominus h) \mid h \in H\}$  in  $L^2(G)$ , then

$$m(0) = 1 \quad \text{and} \quad \sum_{l=0}^{p-1} |m(\omega \oplus \delta_l)|^2 = 1, \quad \omega \in G^*. \quad (7)$$

From this it follows that the equalities

$$b_0 = 1, \quad |b_j|^2 + |b_{j+p^{n-1}}|^2 + \cdots + |b_{j+(p-1)p^{n-1}}|^2 = 1, \quad 0 \leq j \leq p^{n-1} - 1, \quad (8)$$

are necessary (but not sufficient, see Example 3 below) for the system  $\{\varphi(\cdot \ominus h) \mid h \in H\}$  to be orthonormal in  $L^2(G)$ . Under which additional conditions the function  $\varphi$  generates a MRA in  $L^2(G)$ ? Theorem 4 below contains the answer to this question.

A compact subset  $E$  of  $G^*$  is said to be *congruent to  $U^*$  modulo  $H^\perp$*  if  $\mu^*(E) = 1$  and, for each  $\omega \in E$ , there is an element  $h^* \in H^\perp$  such that  $\omega \oplus h^* \in U^*$ . Let  $m$  be the mask of equation (1). We say that  $m$  satisfies the *modified Cohen condition*, if there exists a compact subset  $E$  of  $G^*$  containing a neighbourhood of the zero element such that:

- 1)  $E$  congruent to  $U^*$  modulo  $H^\perp$ ;
- 2) the inequality

$$\inf_{j \in \mathbb{N}} \inf_{\omega \in E} |m(B^{-j}\omega)| > 0 \quad (9)$$

is true.

Since  $E$  is compact, we note that if  $m(\theta) = 1$  then there exists a number  $j_0$  such that  $m(B^{-j}\omega) = 1$  for all  $j > j_0$ ,  $\omega \in E$ . Therefore (9) holds if the polynomial  $m(\omega)$  does not vanish on the sets  $B^{-1}(E), \dots, B^{-j_0}(E)$ . Moreover, we can choose  $j_0 \leq p^n$ , because  $m$  is completely defined by the values (4) (and  $m$  is an  $H^\perp$ -periodic function).

**THEOREM 4.** *Suppose that the refinement equation (1) possesses a solution  $\varphi \in L_c^2(G)$  such that  $\widehat{\varphi}(\theta) = 1$  and the corresponding mask  $m$  satisfies conditions (7) Then the following are equivalent :*

- (a)  $\varphi$  generates a MRA in  $L^2(G)$ ;
- (b)  $m$  satisfies the modified Cohen's condition;
- (c)  $m$  has no blocked sets.

The proofs of Theorem 1 - 4 are given by the author in the recent paper [14]; some similar results for the dyadic refinable functions and wavelets on  $\mathbb{R}$  have been



obtained in [11] (see also [18]). For  $p = 2$  the equivalence (a)  $\Leftrightarrow$  (b) of Theorem 4 was found by W. Lang in [9].

EXAMPLE 2. Let  $p = n = 2$  and

$$b_0 = 1, b_1 = a, b_2 = 0, b_3 = b,$$

where  $|a|^2 + |b|^2 = 1$ . Put  $a_0 = (1 + a + b)/4$ ,  $a_1 = (1 + a - b)/4$ ,  $a_2 = (1 - a - b)/4$ ,  $a_3 = (1 - a + b)/4$ .

For  $a \neq 0$  the modified Cohen condition is fulfilled on the set  $E = U^*$  and hence the corresponding solution  $\varphi$  generates a MRA in  $L^2(G)$ . In particular, for  $a = 1$  and  $a = -1$  the Haar function:  $\varphi(x) = \mathbf{1}_U(x)$  and the displaced Haar function:  $\varphi(x) = \mathbf{1}_U(x \ominus h_{[1]})$  are obtained respectively. If  $0 < |a| < 1$ , then a solution  $\varphi$  is defined by the expansion

$$\varphi(x) = (1/2)\mathbf{1}_U(A^{-1}x)(1 + a \sum_{j=0}^{\infty} b^j W_{2^{j+1}-1}(A^{-1}x)), \quad x \in G. \quad (10)$$

In the case  $a = 0$  the set  $U_{1,1}^*$  is a blocked set, a function  $\varphi$  is defined by the formula  $\varphi(x) = (1/2)\mathbf{1}_U(A^{-1}x)$  and the system  $\{\varphi(\cdot \ominus h) \mid h \in H\}$  is linear dependence.

The decomposition (10) was found by W. Lang in [8]. When  $|b| < 1/2$  the corresponding wavelets form an unconditional basis in all spaces  $L^q(G)$ ,  $1 < q < \infty$ . Moreover, the relevant wavelets on the line may be identified as multiwavelets consisting of piecewise fractal functions, in the sense of Massopust; see [9] and [10] for the details.

REMARK 1. In [12], a method for finding estimates of regularity of refinable functions on Vilenkin groups was developed. When  $\varphi$  is given by (10) we have the sharp estimate

$$\sup\{|\varphi(x) - \varphi(y)| \mid x, y \in U_{-1}, x \ominus y \in U_j\} \leq C|b|^j, \quad j \in \mathbb{N}$$

(see Example 4.3 in [12]). Also, it is known that the exponent of regularity of a refinable function for small  $p$  and  $n$  can be computed using the joint spectral radius of some linear finite-dimensional operators which are defined by the coefficients of the corresponding refinement equation (cf. [11], Remark 3, [17]).

REMARK 2. Suppose that  $\varphi$  generates a MRA in  $L^2(G)$ . For each  $j \in \mathbb{Z}$  let us denote by  $P_j$  the orthogonal projection of  $L^2(G)$  on  $V_j$ . If known that a "signal"  $f$  belongs to some class  $\mathcal{M}$  in  $L^2(G)$ , then it is possible to seek the parameters  $b_s$ , which minimize, for some fix  $j$ , the quantity

$$\sup\{\|f - P_j f\| \mid f \in \mathcal{M}\}$$

and to study the behavior of this quantity as  $j \rightarrow +\infty$  (cf. [17]). We refer to [12] and [19] for an adapted multiresolution analysis in  $L^2(G)$  based on the entropy estimates.

#### 4 Expansion in Walsh Series

Assume that a compactly supported solution  $\varphi$  of equation (1) generates a MRA in  $L^2(G)$  and  $\widehat{\varphi}(\theta) = 1$ . Further, suppose that the values of mask (2) on the cosets  $U_{n,s}^*$  satisfy condition (8) and let  $\gamma(i_1, i_2, \dots, i_n) = b_s$ , if

$$s = i_1 p^0 + i_2 p^1 + \dots + i_n p^{n-1}, \quad i_j \in \{0, 1, \dots, p-1\}.$$

Then for an integer  $l$  with the  $p$ -ary expansion

$$l = \sum_{j=0}^k \mu_j p^j, \quad \mu_j \in \{0, 1, \dots, p-1\}, \quad \mu_k \neq 0, \quad k = k(l) \in \mathbb{Z}_+, \quad (11)$$

we define  $c_l[m]$  as follows

$$\begin{aligned} c_l[m] &= \gamma(\mu_0, 0, 0, \dots, 0, 0) \quad \text{if } k(l) = 0; \\ c_l[m] &= \gamma(\mu_1, 0, 0, \dots, 0, 0) \gamma(\mu_0, \mu_1, 0, \dots, 0, 0) \quad \text{if } k(l) = 1; \\ &\dots \\ c_l[m] &= \gamma(\mu_k, 0, 0, \dots, 0, 0) \gamma(\mu_{k-1}, \mu_k, 0, \dots, 0, 0) \\ &\quad \dots \gamma(\mu_0, \mu_1, \mu_2, \dots, \mu_{n-2}, \mu_{n-1}) \quad \text{if } k = k(l) \geq n-1. \end{aligned}$$

The indices of each factor in the last product, starting with the second, are equal to the indices of the preceding factor shifted one position rightwards; at the free first position one puts the corresponding digit of the  $p$ -ary expansion (11).

Let  $\mathbb{N}_0(p, n)$  be the set of all positive integers  $l \geq p^{n-1}$  whose  $p$ -ary expansion (11) contains no  $n$ -tuple  $(\mu_j, \mu_{j+1}, \dots, \mu_{j+n-1})$  coinciding with any of the  $n$ -tuples

$$(0, 0, \dots, 0, 1), (0, 0, \dots, 0, 2), \dots, (0, 0, \dots, 0, p-1).$$

Then  $\varphi$  can be written as the following lacunary Walsh series:

$$\varphi(x) = (1/p^{n-1}) \mathbf{1}_U(A^{1-n}x) \left(1 + \sum_{l \in \mathbb{N}(p, n)} c_l[m] W_l(A^{1-n}x)\right), \quad x \in G, \quad (12)$$

where  $\mathbb{N}(p, n) = \{1, 2, \dots, p^{n-1} - 1\} \cup \mathbb{N}_0(p, n)$  (see [12]). This result seems surprising, since Lang noted in [9] that even for  $p = 2, n = 3$  "no simple patterns appear in the coefficients" in the Walsh expansion of  $\varphi$ . Certainly, in the case  $p = n = 2$  the decompositions (10) and (12) coincide.

## 5 Construction for the Case $p = 3$

Following a standard approach (e.g., [20,21]), we reduce the problem of  $p$ -wavelet decomposition into a problem of matrix extension. More precisely, using Theorem 4 we shall discuss the following *procedure to construct orthogonal  $p$ -wavelets in  $L^2(G)$* :

1. Choose numbers  $b_s$ ,  $0 \leq s \leq p^n - 1$ , so that (8) is true.
2. Compute  $a_\alpha$ ,  $0 \leq \alpha \leq p^n - 1$ , by (3) and verify that the mask

$$m_0(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha \overline{W_\alpha^*(\omega)}.$$

has no blocked sets.

3. Find

$$m_l(\omega) = \sum_{\alpha \in \mathbb{Z}_+} a_\alpha^{(l)} \overline{W_\alpha^*(\omega)}, \quad 1 \leq l \leq p-1,$$

such that  $M(\omega) := (m_l(\omega \oplus B^{-1}\omega_{[k]}))_{l,k=0}^{p-1}$  is an unitary matrix.

4. Define  $\psi_1, \dots, \psi_{p-1}$  by the formula

$$\psi_l(x) = p \sum_{\alpha \in \mathbb{Z}_+} a_\alpha^{(l)} \varphi(Ax \ominus h_{[\alpha]}), \quad 1 \leq l \leq p-1.$$

In the  $p = 2$  case one can choose  $a_\alpha^{(1)} = (-1)^\alpha a_{\alpha \oplus 1}$  or  $a_\alpha^{(1)} = (-1)^\alpha a_{2^n-1-\alpha}$  for  $0 \leq \alpha \leq 2^n - 1$  (and  $a_\alpha^{(1)} = 0$  for the rest  $\alpha$ ); cf. [9], [11].

In the the  $p > 2$  case we take the coefficients  $a_\alpha$  as in Step 2 (so that  $b_s$  satisfy (8) and  $m_0$  has no blocked sets). Then

$$\sum_{\alpha=0}^{p^n-1} |a_\alpha|^2 = \frac{1}{p}. \quad (13)$$

In fact, Parseval's relation for the discrete transforms (3) and (4) can be written as

$$\sum_{\alpha=0}^{p^n-1} |a_\alpha|^2 = \frac{1}{p^n} \sum_{\alpha=0}^{p^n-1} |b_\alpha|^2.$$

Therefore (13) follows from (8). Now we define

$$A_{0k}(z) = \sum_{l=0}^{p^{n-1}-1} a_{k+pl} z^l, \quad 0 \leq k \leq p-1,$$

and introduce the polynomials  $A_{lk}(z)$ ,  $\deg A_{lk} \leq p^{n-1} - 1$ , such that

$$m_l(\omega) = \sum_{k=0}^{p-1} \overline{W_k^*(\omega)} A_{lk}(\overline{W_p^*(\omega)}), \quad 1 \leq l \leq p-1. \quad (14)$$

It follows immediately that

$$M(\omega) = A(\overline{W_p^*(\omega)})W^*(\omega), \quad (15)$$

where  $A(z) := (A_{lk}(z))_{l,k=0}^{p-1}$ ,  $W^*(\omega) := (W_l^*(\omega \oplus B^{-1}\omega_{[k]}))_{l,k=0}^{p-1}$ . The matrix  $p^{-1/2}W^*(\omega)$  is unitary. Thus, by (15), unitarity of  $M(\omega)$  is equivalent to that of the matrix  $p^{-1/2}A(z)$  with  $z = \overline{W_p^*(\omega)}$ . From this we claim that Step 3 of the procedure can be realized by some modification of the algorithm for matrix extension suggested by W. Lawton, S.L. Lee and Zuowei Shen in [22] (see also [23], Theorem 2.1).

We illustrate the described procedure by the following example.

EXAMPLE 3. Let  $p = 3$ ,  $n = 2$  and  $b_0 = 1$ ,  $b_1 = a$ ,  $b_2 = \alpha$ ,  $b_3 = 0$ ,  $b_4 = b$ ,  $b_5 = \beta$ ,  $b_6 = 0$ ,  $b_7 = c$ ,  $b_8 = \gamma$ , where

$$|a|^2 + |b|^2 + |c|^2 = |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1.$$

Then (3) implies precisely that

$$\begin{aligned} a_0 &= \frac{1}{9}(1 + a + b + c + \alpha + \beta + \gamma), \\ a_1 &= \frac{1}{9}(1 + a + \alpha + (b + \beta)\varepsilon_3^2 + (c + \gamma)\varepsilon_3), \\ a_2 &= \frac{1}{9}(1 + a + \alpha + (b + \beta)\varepsilon_3 + (c + \gamma)\varepsilon_3^2), \\ a_3 &= \frac{1}{9}(1 + (a + b + c)\varepsilon_3^2 + (\alpha + \beta + \gamma)\varepsilon_3), \\ a_4 &= \frac{1}{9}(1 + c + \beta + (a + \gamma)\varepsilon_3^2 + (b + \alpha)\varepsilon_3), \\ a_5 &= \frac{1}{9}(1 + b + \gamma + (a + \beta)\varepsilon_3^2 + (c + \alpha)\varepsilon_3), \\ a_6 &= \frac{1}{9}(1 + (a + b + c)\varepsilon_3 + (\alpha + \beta + \gamma)\varepsilon_3^2), \\ a_7 &= \frac{1}{9}(1 + b + \gamma + (a + \beta)\varepsilon_3 + (c + \alpha)\varepsilon_3^2), \\ a_8 &= \frac{1}{9}(1 + c + \beta + (a + \gamma)\varepsilon_3 + (b + \alpha)\varepsilon_3^2), \end{aligned}$$

where  $\varepsilon_3 = \exp(2\pi i/3)$ . Further, if  $\gamma(1,0) = a$ ,  $\gamma(2,0) = \alpha$ ,  $\gamma(1,1) = b$ ,  $\gamma(2,1) = \beta$ ,  $\gamma(1,2) = c$ ,  $\gamma(2,2) = \gamma$ , and  $\mu_j \in \{1,2\}$ , then we let:

$$\begin{aligned} c_l[m] &= \gamma(\mu_0, 0) \quad \text{for } l = \mu_0; \\ c_l[m] &= \gamma(\mu_1, 0)\gamma(\mu_0, \mu_1) \quad \text{for } l = \mu_0 + 3\mu_1; \\ &\dots \\ c_l[m] &= \gamma(\mu_k, 0)\gamma(\mu_{k-1}, \mu_k) \dots \gamma(\mu_0, \mu_1) \quad \text{for } l = \sum_{j=0}^k \mu_j 3^j, \quad k \geq 2. \end{aligned}$$

According to (12), we get

$$\varphi(x) = (1/3)\mathbf{1}_U(A^{-1}x)(1 + \sum_l c_l[m]W_l(A^{-1}x)), \quad x \in G.$$

The blocked sets are:

- 1)  $U_{1,1}^*$  for  $a = c = 0$ ,
- 2)  $U_{1,2}^*$  for  $\alpha = \beta = 0$ ,
- 3)  $U_{1,1}^* \cup U_{1,2}^*$  for  $a = \alpha = 0$ .

Hence,  $\varphi$  generates a MRA in  $L^2(G)$  in the following cases:

- 1)  $a \neq 0, \alpha \neq 0$ ,
- 2)  $a = 0, \alpha \neq 0, c \neq 0$ ,
- 3)  $\alpha = 0, a \neq 0, \beta \neq 0$ .

By the definition of  $m_0$  we have

$$m_0(\omega) = A_{00}(\overline{W_3^*(\omega)}) + \overline{W_1^*(\omega)}A_{01}(\overline{W_3^*(\omega)}) + \overline{W_2^*(\omega)}A_{02}(\overline{W_3^*(\omega)}),$$

where  $A_{00}(z) = a_0 + a_3z + a_6z^2$ ,  $A_{01}(z) = a_1 + a_4z + a_7z^2$ ,  $A_{02}(z) = a_2 + a_5z + a_8z^2$ . Now, we require

$$a \neq 0, \quad \alpha = \bar{a}, \quad a\bar{\alpha} + b\bar{\beta} + c\bar{\gamma} = \bar{a}. \quad (16)$$

In particular, for  $0 < a < 1$  one can choose numbers  $\theta, t$  such that

$$\cos(\theta - t) = \frac{a}{1+a}$$

and then set  $\alpha = a$ ,  $r = \sqrt{1-a^2}$ ,  $\beta = r \cos \theta$ ,  $\gamma = r \sin \theta$ ,  $b = r \cos t$ ,  $c = r \sin t$ .

Under the assumptions (16) the mask  $m_0$  has no blocked sets. Moreover, it follows from (13) and (16) that

$$|A_{00}(z)|^2 + |A_{01}(z)|^2 + |A_{02}(z)|^2 = \frac{1}{3}$$

for all  $z$  on the unit circle  $\mathbb{T}$ . To see this, note that by a direct calculation

$$\begin{aligned} |A_{00}(z)|^2 + |A_{01}(z)|^2 + |A_{02}(z)|^2 &= \sum_{k=0}^8 |a_k|^2 + 2\operatorname{Re}[(a_0\bar{a}_3 + a_1\bar{a}_4 + a_2\bar{a}_5)z] \\ &\quad + 2\operatorname{Re}[(a_0\bar{a}_6 + a_1\bar{a}_7 + a_2\bar{a}_8)z^2] + 2\operatorname{Re}[(a_3\bar{a}_6 + a_4\bar{a}_7 + a_5\bar{a}_8)z\bar{z}^2], \end{aligned}$$

where

$$\begin{aligned} &27(a_0\bar{a}_3 + a_1\bar{a}_4 + a_2\bar{a}_5) \\ &\quad = a + \alpha + (\bar{\alpha} + a\bar{\alpha} + b\bar{\beta} + c\bar{\gamma})\varepsilon_3 + (\bar{a} + \bar{a}\alpha + \bar{b}\beta + \bar{c}\gamma)\varepsilon_3^2, \\ &27(a_0\bar{a}_6 + a_1\bar{a}_7 + a_2\bar{a}_8) \\ &\quad = a + \alpha + (\bar{a} + \bar{a}\alpha + \bar{b}\beta + \bar{c}\gamma)\varepsilon_3 + (\bar{\alpha} + a\bar{\alpha} + b\bar{\beta} + c\bar{\gamma})\varepsilon_3^2, \\ &27(a_3\bar{a}_6 + a_4\bar{a}_7 + a_5\bar{a}_8) \\ &\quad = 2\varepsilon_3\operatorname{Re} a + 2\varepsilon_3^2\operatorname{Re} \alpha + 2\operatorname{Re}(a\bar{\alpha} + b\bar{\beta} + c\bar{\gamma}). \end{aligned}$$

Further, if

$$\alpha_0 = \sqrt{3}(a_0, a_1, a_2), \quad \alpha_1 = \sqrt{3}(a_3, a_4, a_5), \quad \alpha_2 = \sqrt{3}(a_6, a_7, a_8),$$

then

$$|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 = 1, \quad \langle \alpha_0, \alpha_1 \rangle = \langle \alpha_0, \alpha_2 \rangle = \langle \alpha_1, \alpha_2 \rangle = 0,$$

where  $\langle \cdot, \cdot \rangle$  is the inner product in  $\mathbb{C}^3$ . It is clear that

$$\alpha_0 + \alpha_1 z + \alpha_2 z^2 = \sqrt{3}(A_{00}(z), A_{01}(z), A_{02}(z)).$$

Let  $P_2$  be the orthogonal projection onto  $\alpha_2$ , i.e.,

$$P_2 w = \frac{\langle w, \alpha_2 \rangle}{\langle \alpha_2, \alpha_2 \rangle} \alpha_2, \quad w \in \mathbb{C}^3.$$

Then we have

$$(I - P_2 + z^{-1}P_2)(\alpha_0 + \alpha_1 z + \alpha_2 z^2)$$

$$= (I - P_2)\alpha_0 + P_2\alpha_1 + z(P_2\alpha_2 + (I - P_2)\alpha_1) =: \beta_0 + \beta_1 z.$$

One now verifies that

$$|\beta_0|^2 + |\beta_1|^2 = 1, \quad \langle \beta_0, \beta_1 \rangle = 0.$$

Futhermore, if  $P_1$  is the orthogonal projection onto  $\beta_1$ , then

$$(I - P_1 + z^{-1}P_1)(\beta_0 + \beta_1 z) = (I - P_1)\beta_0 + P_1\beta_1 =: \gamma_0.$$

By the Gram-Schmidt orthogonalization, we can find an unitary matrix  $\Gamma_0$  once the first row of this matrix is the unit vector  $\gamma_0$ . Then we set

$$\Gamma_1(z) = (I - P_1 + zP_1)\Gamma_0 \quad \text{and} \quad \Gamma_2(z) = (I - P_2 + zP_2)\Gamma_1(z).$$

The first row of  $\Gamma_2(z)$  coincides with  $\alpha_0 + \alpha_1 z + \alpha_2 z^2$ . Putting

$$(A_{lk}(z))_{l,k=0}^2 = \frac{1}{\sqrt{3}}\Gamma_2(z),$$

we see that  $m_1$  and  $m_2$  can be defined as follows:

$$m_l(\omega) = \sum_{k=0}^2 \overline{W_k^*(\omega)} A_{lk}(\overline{W_3^*(\omega)}) = \sum_{\alpha=0}^8 a_\alpha^{(l)} \overline{W_\alpha^*(\omega)}, \quad l = 1, 2.$$

Finally, we find

$$\psi_l(x) = 3 \sum_{\alpha=0}^8 a_\alpha^{(l)} \varphi(Ax \ominus h_{[\alpha]}), \quad l = 1, 2.$$

REMARK 3. For any  $n, p$  we have

$$m_0(\omega) = \sum_{k=0}^{p-1} \overline{W_k^*(\omega)} A_{0k}(\overline{W_p^*(\omega)}).$$

If we require

$$\sum_{k=0}^{p-1} |A_{0k}(z)|^2 = \frac{1}{p} \quad \text{for all } z \in \mathbb{T}, \quad (17)$$

then the vectors

$$\alpha_l = \sqrt{p}(a_{pl}, a_{pl+1}, \dots, a_{pl+p-1}), \quad 0 \leq l \leq p-1,$$

form an orthonormal basis in  $\mathbb{C}^p$ . In this case Step 3 of the procedure can be realized as in Example 3. However, it is hard to use known methods of matrix extension to construct  $\psi_1, \dots, \psi_{p-1}$  without the assumption (17).

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